

Volume and rigidity of hyperbolic polyhedral 3-manifolds

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Abstract

We investigate the rigidity of hyperbolic cone metrics on 3-manifolds which are isometric gluing of ideal and hyper-ideal tetrahedra in hyperbolic spaces. These metrics will be called ideal and hyper-ideal hyperbolic polyhedral metrics. It is shown that a hyper-ideal hyperbolic polyhedral metric is determined up to isometry by its curvature and a decorated ideal hyperbolic polyhedral metric is determined up to isometry and change of decorations by its curvature. The main tool used in the proof is the Fenchel dual of the volume function.

1 Introduction

1.1 Statements of results

We study geometry of 3-dimensional spaces which are isometric gluing of (ideal and hyper-ideal) tetrahedra in hyperbolic spaces. Our main focus is on the rigidity of these spaces. The metrics of these spaces are given by the lengths of edges of tetrahedra (in the underlying triangulation). The curvatures of the spaces are 2π less the cone angles at the edges. Our main results state that for a fixed triangulation, the curvature determines the edge lengths and hence these hyperbolic polyhedral metrics.

The tool used in the proof is a variational principle associated to the Schläfli formula and its Legendre transformation. The infinitesimal rigidity of hyperbolic cone metrics follows from the strict convexity of the volume of the ideal and hyper-ideal tetrahedra in terms of the dihedral angles. In the dual setting, one considers the co-volume which is the dual of the volume of the tetrahedra and has the edge lengths as the variables. The main difficulty comes from the fact that the space of hyperbolic tetrahedra parametrized by the edge lengths is not convex. We overcome the difficulty by showing that the co-volume function (of the edge lengths) can be extended to a C^1 smooth convex function defined on a convex open set. This is very similar to the results established in [23]. By establishing the convex extensions of the co-volume functions, we are able to prove several results on the volume optimization program of Casson and Rivin. For instance, we show that the maximum volume angle structures are exactly those coming from the generalized polyhedral metrics (see theorem 1.3).

We now state our results more precisely. Suppose (M, \mathcal{T}) is a triangulated compact pseudo 3-manifold with a triangulation \mathcal{T} and the set of edges $E = E(\mathcal{T})$.

Definition 1.1. A decorated hyperbolic polyhedral metric (*respectively* hyper-ideal polyhedral metric) on (M, \mathcal{T}) is obtained by replacing each tetrahedra in \mathcal{T} by an decorated ideal tetrahedron (*respectively* hyper-ideal tetrahedron) and replacing the affine gluing homeomorphisms by isometries preserving the decoration. The curvature of the metric assign each edge e 2π less the cone angle at e for interior edge e and π less the cone angle for boundary edge e .

By the construction, these polyhedral metrics are determined by the lengths of the edges.

Theorem 1.2. Suppose (M, \mathcal{T}) is a triangulated compact pseudo 3-manifold (M, \mathcal{T}) .

- (a) A decorated hyperbolic polyhedral metric on (M, \mathcal{T}) is determined up to isometry and change of decorations by its curvature.
- (b) A hyper-ideal hyperbolic polyhedral metric on (M, \mathcal{T}) is determined up to isometry by its curvature.

The rigidity results are closely related to the volume optimization of angle structures initiated by Casson and Rivin. The program tries to find complete hyperbolic metrics on ideal triangulated 3-manifolds (N, \mathcal{T}) using *angle structures* (see [20], [33]). Recall that a non-negative (respectively positive) angle structure α on (N, \mathcal{T}) assigns each edge in a tetrahedron a non-negative (respectively positive) number called the angle so that the sum of angles around each edge is 2π and the sum of angles at three edges from each vertex of each tetrahedron is π . The *volume* of an angle structure α is well defined using the Lobachevsky function (see §2.1). It is well known there are maximum volume non-negative angle structures. If the maximum volume angle structure is positive, then it is known [15, 5, 33] that there exists a geometric triangulation of a complete hyperbolic metric on the manifold $N - \{\text{vertices}\}$ realizing the angle structure. Our result gives a characterization of maximum angle structures in the case some angles are 0.

Theorem 1.3. *Suppose (N, \mathcal{T}) is a triangulated closed pseudo 3-manifold which supports a positive angle structure and α is a non-negative angle structure which maximizes volume in the space of all non-negative angle structures on (N, \mathcal{T}) . Then there exists an assignment of real number $l(e)$ to each edge e so that for each tetrahedron $\sigma \in \mathcal{T}$,*

(1) if all angles of σ in α are positive, then α are the dihedral angles of the decorated ideal tetrahedron of edge lengths given by l and,

(2) if one angle of σ in α is 0, then all angles of σ in α are $0, 0, 0, 0, \pi, \pi$ and their edge lengths in l satisfy

$$e^{\frac{l_1+l_4}{2}} \geq e^{\frac{l_2+l_5}{2}} + e^{\frac{l_3+l_6}{2}} \quad (1.1)$$

where l_1 is the length of the edge of angle π and l_i and l_{i+3} are lengths of opposite edges.

Conversely, if $l : E \rightarrow \mathbb{R}$ is any function so that (1) and (2) hold, then the corresponding angle α of l defined by (1) and (2) maximizes volume.

In section 6, we introduce the corresponding notion of non-negative and positive angle structures of hyper-ideal type (see Definition 6.1), and prove the following counterpart of theorem 1.3.

Theorem 1.4. *Suppose (N, \mathcal{T}) is a triangulated closed pseudo 3-manifold which supports a positive angle structure of hyper-ideal type and α maximizes the volume in the space of all non-negative angle structures of hyper-ideal type on (N, \mathcal{T}) . Then there exists an assignment of positive number $l(e)$ to each edge e so that for each tetrahedron $\sigma \in \mathcal{T}$,*

(1) if all angles of σ in α are positive, then α are the dihedral angles of the hyper-ideal tetrahedron of edge lengths given by l and,

(2) if one angle of σ in α is 0, then all angles of σ in α are $0, 0, 0, 0, \pi, \pi$ and the numbers assigned by l to the edges of σ are not the edge lengths of any hyper-ideal tetrahedron.

Conversely, if $l : E \rightarrow \mathbb{R}_{>0}$ is any function so that (1) and (2) holds, then the corresponding angle α of l defined by (1) and (2) maximizes volume.

There have been many important work on rigidity of hyperbolic cone metrics on 3-manifolds. See work of Hodgson-Kerckhoff [17], Weiss [40], Mazzeo-Montcouquiol [27, 28] and Fillastre-Izmestiev [11, 12, 18] and others. The difference between their work and ours is that we consider the case where the singularity consists of complete geodesics from cusp to cusp or geodesics orthogonal to the totally geodesic boundary with possible cone singularities.

The paper is organized as follows. In Sections 2, we collect preliminary materials including decorated hyperbolic tetrahedra, angle structures, volume functions and the Fenchel dual. In Section 3, we define the co-volume function and reveal its relationship with the Fenchel dual of the volume function. As a consequence, we prove Theorem 1.2 (a) and Theorem 1.3. The second part of the paper focuses on polyhedral metrics and angle structures of hyper-ideal type. The corresponding preliminary materials are included in Section 4, and Theorem 1.2 (b) and Theorem 1.4 are respectively proved in Section 5 and Section 6.

The work is partially supported by the NSF DMS 1105808 and NSF DMS 1222663. We would like to thank Steven Kerckhoff for discussions.

2 Preliminaries on triangulations, volume and Fenchel duality

Since this paper involves topological triangulations, geometry of tetrahedra in hyperbolic space \mathbb{H}^3 and convex optimization, we will briefly recall the related material in this section.

2.1 Triangulations

Take a finite disjoint collection T of Euclidean tetrahedra and identify some of the codimension-1 faces in T in pairs by affine homeomorphisms. The quotient space (M, \mathcal{T}) is a compact pseudo 3-manifold M together with a triangulation \mathcal{T} whose simplexes are the quotients of simplexes in T . If each codimension-1 face of T are identified with another codimension-1 face, then M is a closed pseudo 3-manifold. Otherwise, M is a compact pseudo 3-manifold with non-empty boundary which is the quotient of the union of un-identified codimension-1 faces in T .

Two edges of tetrahedra in T are called equivalent if they are mapped to the same set in M . We define *edges* in the triangulation \mathcal{T} to be equivalence classes of edges in tetrahedra in T . We use $E = E(\mathcal{T})$ and $T(\mathcal{T})$ to denote the sets of all edges and tetrahedra in \mathcal{T} respectively. Since a tetrahedron in the triangulation \mathcal{T} is the same as a tetrahedron in the original set T , we will identify $T(\mathcal{T})$ with the set T . A *quad* in the triangulation \mathcal{T} is a pair of opposite edges in T . Thus each tetrahedron contains three quads. We use $\square = \square(\mathcal{T})$ to denote the set of all quads in \mathcal{T} . If $q \in \square$, $e \in E$ and $\sigma \in T$, we use $q \subset \sigma$ to denote that the quad q is contained in the tetrahedron σ and use $q \sim e$ or $e \sim q$ to denote that $q \cap e = \emptyset$ and there exists $\sigma \in T$ with $q \subset \sigma$ and $\sigma \cap e \neq \emptyset$.

Using these notations and the fact that angles at opposite edges in a tetrahedron are the same for any angle structure, a *non-negative angle structure* on a closed pseudo 3-manifold (M, \mathcal{T}) is a map $x : \square \rightarrow \mathbb{R}_{\geq 0}$ so that (1) $\forall \sigma \in T$, $\sum_{q \subset \sigma} x(q) = \pi$ and (2) $\forall e \in E$, $\sum_{q \sim e} x(q) = 2\pi$. See for instance [25] for more details.

2.2 Decorated ideal tetrahedra in the hyperbolic 3-space

An *ideal n-simplex* s in the hyperbolic n -space \mathbb{H}^n is the convex hull of $n+1$ points v_1, \dots, v_{n+1} in $\partial\mathbb{H}^n$ so that $\{v_1, \dots, v_{n+1}\}$ are not in a round $(n-1)$ -sphere. Any two ideal triangles are isometric. An ideal tetrahedron in \mathbb{H}^3 is determined up to isometry by its six dihedral angles. These angles satisfy the condition that angles at opposite edges are the same and the sum of all angles is 2π . Thus the space of all ideal tetrahedra modulo isometry can be identified with $\mathcal{A} = \{(a, b, c) \in \mathbb{R}_{>0}^3 | a + b + c = \pi\}$.

Following Penner [30], a *decorated ideal n-simplex* (or simply decorated simplex) is a pair $(s, \{H_1, \dots, H_{n+1}\})$ where s is an ideal n -simplex and H_i is an $(n-1)$ -horosphere centered at the i -th vertex v_i . We call $\{H_1, \dots, H_{n+1}\}$ the *decoration*. Two decorated simplexes are *equivalent* if there is a decoration preserving isometry between the underlying ideal simplexes. Each edge $e = v_i v_j$ in a decorated simplex has the signed *length* l_{ij} defined as follows. The absolute value $|l_{ij}|$ of the length is the distance between

$H_i \cap e$ and $H_j \cap e$ so that $l_{ij} > 0$ if H_i and H_j are disjoint and $l_{ij} \leq 0$ if $H_i \cap H_j \neq \emptyset$. Faces of decorated ideal simplexes are decorated ideal simplexes. Also $s \cap H_i$ is isometric to a Euclidean $(n-1)$ -simplex.

For a decorated ideal triangle $(s, \{H_1, H_2, H_3\})$, we call the length a_i of the horocyclic arc in H_i bounded by the two edges of s from v_i the *angle* at v_i . Penner's cosine law says that for $\{i, j, k\} = \{1, 2, 3\}$,

$$a_i = e^{\frac{l_{jk} - l_{ij} - l_{ik}}{2}}. \quad (2.1)$$

Given any three real numbers l_1, l_2, l_3 , there exists a unique decorated ideal triangle whose lengths are l_1, l_2, l_3 . See [30].

The characterization of the lengths of decorated ideal tetrahedron is well known (see for instance [24] lemma 2.5, or [4] lemma 4.2.3).

Lemma 2.1. *Suppose $\{l_{ij}\}$ are the edge lengths of a decorated ideal tetrahedron $\sigma = (s, \{H_1, \dots, H_4\})$. Then all four Euclidean triangles $\{H_i \cap s\}$ are similar to the Euclidean triangle τ of edge lengths $e^{\frac{l_{ij} + l_{kh}}{2}}$, so that*

$$e^{\frac{l_{ij} + l_{kh}}{2}} + e^{\frac{l_{ik} + l_{jh}}{2}} > e^{\frac{l_{ih} + l_{jk}}{2}}$$

for $\{i, j, k, h\} = \{1, 2, 3, 4\}$. The dihedral angle α_{ij} of σ at the edge $v_i v_j$ is equal to the inner angle of τ opposite to the edge of length $e^{\frac{l_{ij} + l_{kh}}{2}}$. Conversely, if $(l_{12}, \dots, l_{34}) \in \mathbb{R}^6$ satisfies the triangular inequalities above, then there is a unique decorated ideal tetrahedron having l_{ij} as the length at the edge $v_i v_j$.

One consequence of the lemma is that dihedral angles $\alpha_{ij} = \alpha_{kl}$, i.e., dihedral angles at opposite edges are the same. Thus, we can talk about the dihedral angle of a quad in a decorated simplex.

2.3 Generalized decorated tetrahedra, dihedral angles and volume

A *generalized decorated tetrahedron* is a (topological) 3-simplex of vertices v_1, \dots, v_4 so that each edge $v_i v_j$ is assigned a real number $l_{ij} = l_{ji}$, called the *length*. A decorated ideal tetrahedron (with the signed edge lengths) is a generalized decorated tetrahedron. The space of all generalized decorated tetrahedra parameterized by the length vectors $l = (l_{12}, \dots, l_{34})$ is \mathbb{R}^6 . The subspace of all (equivalence classes) of decorated ideal tetrahedra is given by $\{(l_{12}, \dots, l_{34}) \in \mathbb{R}^6 \mid e^{\frac{l_{ij} + l_{kh}}{2}} + e^{\frac{l_{ik} + l_{jh}}{2}} > e^{\frac{l_{ih} + l_{jk}}{2}}, \{i, j, k, h\} \text{ distinct}\}$.

To define *dihedral angles* and *volume* of a generalized decorated tetrahedron σ , let us begin with the notion of *generalized Euclidean triangles* and their *angles*. A *generalized Euclidean triangle* Δ is a (topological) triangle of vertices v_1, v_2, v_3 so that each edge is assigned a positive number, called edge length. Let x_i be the assigned length of the edge $v_j v_k$ where $\{i, j, k\} = \{1, 2, 3\}$. The *inner angle* $a_i = a_i(x_1, x_2, x_3)$ at the vertex v_i is defined as follows. If x_1, x_2, x_3 satisfy the triangle inequalities that $x_j + x_k > x_i$ for $\{h, j, k\} = \{1, 2, 3\}$, then a_i is the inner angle of the Euclidean triangle of edge lengths x_1, x_2, x_3 opposite to the edge of length x_i ; if $x_i \geq x_j + x_k$, then $a_i = \pi$, $a_j = a_k = 0$. It is known (see for instance [23]) that

Lemma 2.2. *The angle function $a_i(x_1, x_2, x_3) : \mathbb{R}_{>0}^3 \rightarrow [0, \pi]$ is continuous so that $a_1 + a_2 + a_3 = \pi$ and the C^0 -smooth differential 1-form $\sum_{i=1}^3 a_i d(\ln x_i)$ is closed on $\mathbb{R}_{>0}^3$. Furthermore, for $u_i = \ln x_i$, the integral $F(u) = \int_0^u \sum_{i=1}^3 a_i(u) du_i$ is a C^1 -smooth convex function in (u_1, u_2, u_3) on \mathbb{R}^3 so that F is strictly convex when restricted to $\{u \in \mathbb{R}^3 \mid u_1 + u_2 + u_3 = 0, e^{u_i} + e^{u_j} > e^{u_k}\}$ and $F(u + (k, k, k)) = F(u) + k\pi$ for all $k \in \mathbb{R}$.*

Here a C^0 -smooth 1-form is defined to be closed if its integration over any C^1 -smooth null homotopic loop is zero.

For a generalized decorated tetrahedron of length vector $l = (l_{12}, \dots, l_{34}) \in \mathbb{R}^6$, the *dihedral angle* $\alpha_{ij} = \alpha_{ij}(l)$ at the edge $v_i v_j$ is defined to be the inner angle of the generalized Euclidean triangle of edge lengths $e^{\frac{l_{ij}+l_{hk}}{2}}$, $e^{\frac{l_{ij}+l_{jh}}{2}}$ and $e^{\frac{l_{ih}+l_{jk}}{2}}$ so that α_{ij} is opposite to the edge of length $e^{\frac{l_{ij}+l_{hk}}{2}}$ for h, i, j, k distinct. In particular, the dihedral angles at opposite edges are the same and the total sum of all six dihedral angles are 2π . If σ is a decorated ideal tetrahedron, then the two definitions of dihedral angles coincide.

Recall that the Lobachevsky function is defined by $\Lambda(x) = -\int_0^x \ln |2 \sin t| dt$. It is a continuous function of period π so that $\Lambda(-x) = -\Lambda(x)$. The *volume* of a generalized decorated simplex σ of lengths l_{ij} 's, denoted by $\text{vol}(l)$, is defined to be $\frac{1}{2} \sum_{i < j} \Lambda(\alpha_{ij}(l))$. If the generalized decorated tetrahedron σ is a decorated ideal tetrahedron, then Lobachevsky showed that $\text{vol}(l)$ is the hyperbolic volume of the underlying ideal tetrahedron. If σ is not a decorated ideal tetrahedron, then by definition, $\text{vol}(l) = \Lambda(0) + \Lambda(0) + \Lambda(\pi) = 0$.

Since the space of all ideal tetrahedra can be parameterized by $\mathcal{A} = \{(a, b, c) \in \mathbb{R}_{>0}^3 \mid a+b+c = \pi\}$, the volume function vol defined on \mathcal{A} is given by $\text{vol}(a, b, c) = \Lambda(a) + \Lambda(b) + \Lambda(c)$. By lemma 2.2, we have

Lemma 2.3. (1) The function $\alpha_{ij} : \mathbb{R}^6 \rightarrow \mathbb{R}$ is continuous.

(2)(Rivin) The volume function $\text{vol} : \mathcal{A} = \{(x, y, z) \in \mathbb{R}_{>0}^3 \mid x + y + z = \pi\} \rightarrow \mathbb{R}$ is smooth strictly concave and extends continuously to the closure $\overline{\mathcal{A}} = \{(x, y, z) \in \mathbb{R}_{\geq 0}^3 \mid x + y + z = \pi\}$ so that $\text{vol}(x, y, z) = 0$ if one of x, y, z is 0.

(3) The volume function on the space of generalized decorated tetrahedra \mathbb{R}^6 , $\text{vol} : \mathbb{R}^6 \rightarrow [0, \infty)$, is continuous.

2.4 Fenchel Duality and hyperbolic volume

Recall that a *proper* convex function $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$ is a convex function so that $f(a) \neq \infty$ for some $a \in \mathbb{R}^n$. A function $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$ is called *lower semi-continuous* if for all $a \in \mathbb{R}^n$, $\liminf_{x \rightarrow a} f(x) \geq f(a)$. If $X \subset \mathbb{R}^n$ is a non-empty closed convex set and $g : X \rightarrow \mathbb{R}$ is a convex lower semi-continuous function, then the new function $\phi_g : \mathbb{R}^n \rightarrow (-\infty, \infty]$ defined by $\phi_g|_X = g$ and $\phi_g(x) = \infty$ for $x \notin X$ is a proper lower semi-continuous convex function. See for instance the classical book [32] for details.

If $u, v \in \mathbb{R}^n$, we use $u \cdot v$ or $\langle u, v \rangle$ to denote the standard inner product of u and v .

Definition 2.4. The Fenchel dual $f^* : \mathbb{R}^n \rightarrow (-\infty, \infty]$ of a proper function $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$ is

$$f^*(y) = \sup\{x \cdot y - f(x) \mid x \in \mathbb{R}^n\}.$$

It is known that if f is a proper convex function then f^* is a proper convex lower semi-continuous function. A fundamental fact about f^* is the following (see [32]),

Theorem 2.5 (Fenchel). If $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$ is a proper lower semi continuous convex function, then $(f^*)^* = f$.

Since the volume function $\text{vol} : \overline{\mathcal{A}} = \{(a, b, c) \in \mathbb{R}_{\geq 0}^3 \mid a + b + c = \pi\}$ is concave and continuous, we obtain a proper lower semi continuous convex function $\phi : \mathbb{R}^3 \rightarrow (-\infty, \infty]$ defined by $\phi(x) = -\text{vol}(x)$ if $x \in \overline{\mathcal{A}}$ and $\phi(x) = \infty$ if $x \notin \overline{\mathcal{A}}$.

Proposition 2.6. *The Fenchel dual ϕ^* of ϕ is the C^1 -smooth convex function defined by*

$$\phi^*(y) = \sum_{i=1}^3 (\Lambda(a_i) + a_i y_i)$$

where $a_1(y), a_2(y), a_3(y)$ are inner angles of the generalized Euclidean triangle of edge lengths e^{y_1}, e^{y_2} and e^{y_3} so that a_i is opposite to the edge of length e^{y_i} . Furthermore,

$$(1) \frac{\partial \phi^*(x)}{\partial y_i} = a_i(y), \text{ and}$$

(2) if $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}_{>0}^3$ so that $\theta_1 + \theta_2 + \theta_3 = \pi$, then the convex function $\psi_\theta(y) = \phi^*(y) - \sum_{i=1}^3 \theta_i y_i$ satisfies that $\psi_\theta(y + (k, k, k)) = \psi_\theta(y)$ for all $k \in \mathbb{R}$ and $\lim_{\max |y_i - y_j| \rightarrow \infty} \psi_\theta(y) = \infty$.

Proof. For $y \in \mathbb{R}^3$, define $g(x) = g_y(x) := \sum_{i=1}^3 (x_i y_i + \Lambda(x_i)) : \bar{\mathcal{A}} \rightarrow \mathbb{R}$. By definition, $\phi^*(y) = \sup\{x \cdot y - \phi(x) | x \in \mathbb{R}^3\} = \sup\{\sum_{i=1}^3 x_i y_i + \Lambda(x_i) | x \in \bar{\mathcal{A}}\} = \max\{g(x) | x \in \bar{\mathcal{A}}\}$. Let $\Omega = \{(z_1, z_2, z_3) | e^{z_i} + e^{z_j} > e^{z_k}, i, j, k \text{ distinct}\}$. If $y \in \Omega$, then g has a critical point at $(a_1(y), a_2(y), a_3(y))$ in \mathcal{A} . Indeed, for the tangent vector $\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}$ to \mathcal{A} ,

$$\begin{aligned} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)(g) &= y_i - y_j - \ln(2|\sin(x_i)|) + \ln(2|\sin(x_j)|) \\ &= \ln\left(\frac{e^{y_i}}{\sin(x_i)}\right) - \ln\left(\frac{e^{y_j}}{\sin(x_j)}\right) = 0 \end{aligned} \quad (2.2)$$

at $x = a(y)$ due to the sine law. Since g is a strictly concave function in $x \in \mathcal{A}$, it follows that $a = (a_1, a_2, a_3)$ is the unique maximum point of g in \mathcal{A} , i.e.,

$$\phi^*(y) = \max\{g(x) | x \in \bar{\mathcal{A}}\} = \sum_{i=1}^3 (a_i y_i + \Lambda(a_i)).$$

If $y \notin \Delta$, say $e^{y_i} \geq e^{y_j} + e^{y_k}$, then $a_i = \pi, a_j = a_k = 0$. The same calculation above shows that $g(x)$ has no critical points in \mathcal{A} . Therefore $\max\{g(x) | x \in \bar{\mathcal{A}}\} = \max\{g(x) | x \in \partial \bar{\mathcal{A}}\}$. On the other hand $(\Lambda(x_1) + \Lambda(x_2) + \Lambda(x_3))|_{\partial \bar{\mathcal{A}}} = 0$, we obtain $\max\{g(x) | x \in \partial \bar{\mathcal{A}}\} = \max\{\sum_{r=1}^3 x_r y_r | x_r \geq 0, x_1 + x_2 + x_3 = \pi, \text{ and some } x_s = 0\}$. But for $x \in \partial \bar{\mathcal{A}}$, $\sum_{r=1}^3 x_r y_r \leq (\sum_{r=1}^3 x_r) y_i = \pi y_i$ so that equality holds for x with $x_i = \pi$ and $x_j = x_k = 0$. Therefore, $\phi^*(y) = \pi y_i = \sum_{j=1}^3 \Lambda(a_j) + a_j y_j$ due to $a_i = \pi, a_j = a_k = 0$. In particular, we see that ϕ^* is convex and hence continuous on \mathbb{R}^3 .

To show C^1 -smoothness of ϕ^* , we first note that by the sine law for Euclidean triangles of angles a_1, a_2, a_3 and lengths $e^{y_1}, e^{y_2}, e^{y_3}$, $d(\sum_{i=1}^3 \Lambda(a_i)) = -\sum_{i=1}^3 \ln|2\sin(a_i)| da_i = -\sum_{i=1}^3 y_i da_i$. See for instance [33]. Therefore, if $y \in \Omega$, $d\phi^* = \sum_{i=1}^3 (a_i dy_i + y_i da_i) + d(\sum_{i=1}^3 \Lambda(a_i)) = \sum_{i=1}^3 a_i dy_i$, i.e., $\nabla \phi^* = (a_1, a_2, a_3)$. In the open set $\mathcal{U}_i = \{y | e^{y_i} > e^{y_j} + e^{y_k}\}$ we have $\phi^*(y) = \pi y_i$. Hence we also have $\nabla \phi^*(y) = (a_1, a_2, a_3)$ on \mathcal{U}_i . On the other hand, the C^1 -smooth function $F(y)$ defined in lemma 2.2 satisfies $\nabla F = (a_1, a_2, a_3)$ on \mathbb{R}^3 . Therefore these two functions ϕ^* and F have the same gradient on the open dense subset $\Omega \cup \bigcup_{i=1}^3 \mathcal{U}_i$. Since F and ϕ^* are continuous, these two functions defer by a constant. Hence ϕ^* is C^1 -smooth.

To prove the last statement, note that $\psi_\theta(y + (k, k, k)) = \psi_\theta(y)$ and convexity of ψ_θ follow from the definition. Furthermore by lemma 2.2, ψ_θ is strictly convex when restricted to $\{y \in \mathbb{R}^3 | \sum_{i=1}^3 y_i = 0, e^{y_i} + e^{y_j} > e^{y_k}\}$. Let $l = (l_1, l_2, l_3) \in \mathbb{R}^3$ be the vector so that the Euclidean triangle of edge lengths $e^{l_1}, e^{l_2}, e^{l_3}$ has inner angles $\theta_1, \theta_2, \theta_3$ and $\sum_{i=1}^3 l_i = 0$. Then $\nabla \psi_\theta(l) = 0$. Consider the plane $P = \{y \in \mathbb{R}^3 | \sum_{i=1}^3 y_i = 0\}$. The restriction $\psi_\theta|_P$ is a convex function with a minimal point l so that ψ_θ is strictly convex near l . Therefore $\lim_{p \in P, p \rightarrow \infty} \psi_\theta(p) = \infty$. Due to $\psi_\theta(y + (k, k, k)) = \psi_\theta(y)$, this shows, by projecting \mathbb{R}^3 to P , that $\lim_{\max\{|y_i - y_j|\} \rightarrow \infty} \psi_\theta(y) = \infty$. \square

We remark that the function $\sum_{i=1}^3 [\Lambda(a_i(y)) + a_i(y)y_i]$ has appeared before in the work of Cohn-Kenyon-Propp [7] and Bobenko-Pinkahl-Springborn [4]. The proposition is similar to the work of Colin de Verdière [8].

2.5 Generalized decorated metrics, angle assignments and volume

Suppose (M, \mathcal{T}) is a triangulated compact pseudo 3-manifold with the sets of edges $E = E(\mathcal{T})$ and quads $\square = \square(\mathcal{T})$.

Definition 2.7 (Angle assignment). *An angle assignment on (M, \mathcal{T}) is a map $\alpha : \square \rightarrow \mathbb{R}_{\geq 0}$ so that for each tetrahedron $\sigma \in T$, $\sum_{q \subset \sigma} \alpha(q) = \pi$. An angle assignment is called positive if $\alpha(q) > 0$ for all q . The cone angle of α is defined to be $k_\alpha : E \rightarrow \mathbb{R}_{\geq 0}$ where $k_\alpha(e) = \sum_{q \sim e} \alpha(q)$. The volume of α , denoted by $\text{vol}(\alpha)$, is defined to be $\text{vol}(\alpha) = \sum_{q \in \square} \Lambda(\alpha(q))$. Given $k : E \rightarrow \mathbb{R}_{\geq 0}$, the space of all angle assignments of cone angle k is denoted by $\mathcal{A}_k^*(M, \mathcal{T})$ and the space of all positive angle assignments of cone angle k is denoted by $\mathcal{A}_k(M, \mathcal{T})$.*

By definition,

$$\mathcal{A}_k^*(M, \mathcal{T}) = \{\alpha \in \mathbb{R}_{\geq 0}^\square \mid k_\alpha = k, \forall \sigma \in T, \sum_{q \subset \sigma} \alpha(q) = \pi\},$$

and

$$\mathcal{A}_k(M, \mathcal{T}) = \{\alpha \in \mathbb{R}_{> 0}^\square \mid k_\alpha = k, \forall \sigma \in T, \sum_{q \subset \sigma} \alpha(q) = \pi\}.$$

Note that if $\mathcal{A}_k(M, \mathcal{T}) \neq \emptyset$, then the closure of $\mathcal{A}_k(M, \mathcal{T})$ in \mathbb{R}^\square is $\mathcal{A}_k^*(M, \mathcal{T})$. However, it is possible that $\mathcal{A}_k(M, \mathcal{T}) = \emptyset$ and $\mathcal{A}_k^*(M, \mathcal{T}) \neq \emptyset$.

The usual angle structures on closed pseudo 3-manifolds (M, \mathcal{T}) are positive angle assignments of cone angle 2π at each edge.

As a consequence of lemma 2.3, we have

Corollary 2.8. *The volume function $\text{vol} : \mathcal{A}_k^*(M, \mathcal{T}) \rightarrow \mathbb{R}$ is continuous, concave and is smooth strictly concave when restricted to $\mathcal{A}_k(M, \mathcal{T})$.*

Definition 2.9 (Generalized decorated metric). *A generalized decorated metric on (M, \mathcal{T}) is given by a map $l : E(\mathcal{T}) \rightarrow \mathbb{R}$, called the edge length function.*

For each $l \in \mathbb{R}^E$, by replacing each tetrahedron σ in T by a generalized decorated tetrahedron whose edge lengths are given by l , we define the *dihedral angle* of l at an edge e in a tetrahedron $\sigma \supset e$ to be the corresponding dihedral angles in the generalized decorated tetrahedron whose edge lengths are given by l . Since dihedral angles are the same at two opposite edges in a generalized decorated tetrahedron, the dihedral angle $\alpha = \alpha_l$ of the generalized decorated metric l , $\alpha : \square \rightarrow [0, \infty)$ is an angle assignment, i.e., $\forall \sigma \in T$, $\sum_{q \subset \sigma} \alpha(q) = \pi$. The cone angle of α is called the *cone angle* of the metric l . We denote it by k_l , i.e., $k_l = k_{\alpha_l}$.

The *volume* of a generalized decorated metric $l \in \mathbb{R}^E$ is defined to be the volume of its dihedral angle assignment, i.e., $\text{vol}(l) = \sum_{q \in \square} \Lambda(\alpha_l(q))$. The *covolume* of $l \in \mathbb{R}^E$, denoted by $\text{cov}(l)$, is defined to be

$$\text{cov}(l) = 2\text{vol}(l) + l \cdot k_l = 2 \sum_{q \in \square} \Lambda(\alpha_l(q)) + \sum_{e \in E} l(e)k_l(e)$$

where k_l is the cone angle of l and $u \cdot v$ is the standard inner product of $u, v \in \mathbb{R}^E$. By the definition of cone angles, we have

$$\text{cov}(l) = \sum_{\sigma \in T} \sum_{q \subset \sigma} [2\Lambda(\alpha_l(q)) + \alpha_l(q) \sum_{e \sim q} l(e)]. \quad (2.3)$$

Proposition 2.10. *The covolume function defined on the space of all generalized decorated tetrahedra \mathbb{R}^6 is given by*

$$\text{cov}(x_1, \dots, x_6) = 2\phi^*\left(\frac{x_1 + x_4}{2}, \frac{x_2 + x_5}{2}, \frac{x_3 + x_6}{2}\right) \quad (2.4)$$

where x_1, \dots, x_6 are the edge lengths so that x_i and x_{i+3} are lengths of opposite edges. In particular, $\text{cov} : \mathbb{R}^6 \rightarrow \mathbb{R}$ is a C^1 -smooth convex function so that

$$\frac{\partial \text{cov}(x)}{\partial x_i} = \alpha_i \quad (2.5)$$

when α_i is the dihedral angle at the i -th edge.

Proof. For $i = 1, 2, 3$, let $y_i = \frac{x_i + x_{i+3}}{2}$ and $\alpha_1, \alpha_2, \alpha_3$ be the angles of the generalized Euclidean triangle of edge lengths $e^{y_1}, e^{y_2}, e^{y_3}$. Then by proposition 2.6, $\phi^* = \sum_{i=1}^3 [\Lambda(\alpha_i) + \frac{1}{2}\alpha_i(x_i + x_{i+3})]$ and $\frac{\partial \phi^*}{\partial y_i} = \alpha_i$. Now by definition, the dihedral angles of generalized tetrahedron of lengths x_i 's are α_i and $\alpha_{i+3} = \alpha_i$ for $i = 1, 2, 3$. Therefore by (2.3), $\text{cov}(x) = 2 \sum_{i=1}^3 [\Lambda(\alpha_i) + \frac{1}{2}\alpha_i(x_i + x_{i+3})] = 2\phi^*(y)$. Furthermore, for $i = 1, 2, 3$, $\frac{\partial \text{cov}(x)}{\partial x_i} = 2 \cdot \frac{1}{2} \cdot \frac{\partial \phi^*}{\partial y_i} = \alpha_i$. The result also works for $i = 4, 5, 6$ due to $\alpha_{i+3} = \alpha_i$. □

3 Volume maximization and covolume minimization

A *decorated ideal hyperbolic polyhedral metric* on (M, \mathcal{T}) is a generalized metric $l \in \mathbb{R}^E$ so that each tetrahedron $\sigma \in T$ becomes a decorated ideal hyperbolic tetrahedron in the length l . These metrics are the same as complete finite volume hyperbolic cone metrics on $M - V(\mathcal{T})$ which are obtained as isometric gluing of ideal tetrahedra along codimension-1 faces together with a horosphere centered at each cusp. The collection of the horospheres is called a *decoration*.

We will establish the main results for decorated ideal hyperbolic polyhedral metrics in this section. These results imply theorems stated in §1.

Theorem 3.1. *Suppose (M, \mathcal{T}) is a compact triangulated pseudo 3-manifold.*

1. *If $l \in \mathbb{R}^{E(\mathcal{T})}$ is a generalized decorated metric on (M, \mathcal{T}) with dihedral angle $\alpha = \alpha_l \in \mathcal{A}_k^*(M, \mathcal{T})$ of cone angle k , then α is a maximum volume point for vol on $\mathcal{A}_k^*(M, \mathcal{T})$.*
2. *If $k \in \mathbb{R}^{E(\mathcal{T})}$ so that $\mathcal{A}_k(M, \mathcal{T}) \neq \emptyset$ and $\alpha \in \mathcal{A}_k^*(M, \mathcal{T})$ is a maximum volume angle assignment, then there exists a generalized decorated metric $l \in \mathbb{R}^{E(\mathcal{T})}$ so that its dihedral angle function is α . Furthermore, the maximum volume angle point α is unique.*

Theorem 3.2. *A decorated ideal hyperbolic polyhedral metric on (M, \mathcal{T}) is determined up to isometry and change of decoration by its cone angles at edges.*

Theorems 3.1 and 3.2 are consequences of a duality result for volume and covolume to be proved below.

3.1 A combinatorics of triangulations

For a triangulation \mathcal{T} of edges $E = E(\mathcal{T})$ and vertices $V = V(\mathcal{T})$, the vector space $\mathbb{R}^{V(\mathcal{T})}$ acts linearly on $\mathbb{R}^{E(\mathcal{T})}$ by

$$(w + x)(vv') = w(u) + w(v') + x(vv') \quad (3.1)$$

where $w \in \mathbb{R}^{V(\mathcal{T})}$, $x \in \mathbb{R}^{E(\mathcal{T})}$ and the edge vv' has vertices v, v' . We will identify $\mathbb{R}^{V(\mathcal{T})}$ with the linear subspace $\mathbb{R}^{V(\mathcal{T})} + 0$ of $\mathbb{R}^{E(\mathcal{T})}$. For each tetrahedron $\sigma \in T$, let $E(\sigma)$ and $V(\sigma)$ be the sets of edges and vertices in σ so that $\mathbb{R}^{V(\sigma)}$ acts on $\mathbb{R}^{E(\sigma)}$ according to (3.1). Let $L_\sigma : \mathbb{R}^{E(\mathcal{T})} \rightarrow \mathbb{R}^{E(\sigma)}/\mathbb{R}^{V(\sigma)}$ be the composition of the restriction map and quotient map. The following lemma was proved in [29], also see [6] page 1354.

Lemma 3.3 (Neumann). *The kernel of the linear map $\prod_{\sigma \in T} L_\sigma : \mathbb{R}^{E(\mathcal{T})} \rightarrow \prod_{\sigma \in T} \mathbb{R}^{E(\sigma)}/\mathbb{R}^{V(\sigma)}$ is $\mathbb{R}^{V(\mathcal{T})}$. In particular, the induced linear map $\mathbb{R}^{E(\mathcal{T})}/\mathbb{R}^{V(\mathcal{T})} \rightarrow \prod_{\sigma \in T} \mathbb{R}^{E(\sigma)}/\mathbb{R}^{V(\sigma)}$ is injective.*

Indeed, if $x \in \mathbb{R}^{E(\mathcal{T})}$ is in the kernel, then for each tetrahedron σ , we can find a function $f_\sigma : V(\sigma) \rightarrow \mathbb{R}$ so that

$$x(vv') = f_\sigma(v) + f_\sigma(v'). \quad (3.2)$$

The goal is to show that if σ and σ' are two tetrahedra sharing the same vertex v , then $f_\sigma(v) = f_{\sigma'}(v)$. Now if σ and σ' have a common triangle face $t > v$, then the result follows by considering the three equations (3.2) at three vertices of t . The geneneral case follows by producing a sequence of tetrahedra $\sigma_0 = \sigma, \sigma_1, \dots, \sigma_m = \sigma'$ so that σ_i and σ_{i+1} have a common triangular face $t_i > v$.

3.2 The covolume functions

For a triangulated pseudo 3-manifold (M, \mathcal{T}) , define

$$\mathcal{C}(\mathcal{T}) = \{k \in \mathbb{R}^{E(\mathcal{T})} \mid \mathcal{A}_k^*(M, \mathcal{T}) \neq \emptyset\} \quad (3.3)$$

to be the space of all cone angles of angle assignments. For a tetrahedron σ with sets of edges $E(\sigma)$ and vertices $V(\sigma)$, the covolume function $cov_\sigma : \mathbb{R}^{E(\sigma)} \rightarrow \mathbb{R}$ sends $x \in \mathbb{R}^{E(\sigma)}$ to $\sum_{q \subset \sigma} [2\Lambda(\alpha(q)) + \alpha(q) \sum_{e \sim q} x(e)]$ is C^1 -smooth and convex. By proposition 2.10, we have $\frac{\partial cov_\sigma(x)}{\partial x(e)} = \sum_{q \sim e} \alpha(q)$. Furthermore, by lemma 2.2, for any $k \in \mathcal{C}(\sigma)$ and $w \in \mathbb{R}^{V(\sigma)}$, the function $cov_\sigma(x) - x \cdot k$ satisfies

$$cov_\sigma(w + x) - (w + x) \cdot k = cov_\sigma(x) - x \cdot k. \quad (3.4)$$

In particular, cov_σ is a convex function defined on the quotient space $\mathbb{R}^{E(\sigma)}/\mathbb{R}^{V(\sigma)}$. Furthermore, if $k = k_\theta$ so that $\theta(q) > 0$ for all $q \in \square$, then

$$\lim_{x \rightarrow \infty, x \in \mathbb{R}^{E(\sigma)}/\mathbb{R}^{V(\sigma)}} [cov_\sigma(x) - x \cdot k] = +\infty. \quad (3.5)$$

This follows from corollary 2.10 and lemma 2.2. For $k_\theta \in \mathcal{C}(\mathcal{T})$ (i.e., $k_\theta(e) = \sum_{q \sim e} \theta(q)$ for an angle assignment θ), consider the function $cov_{\mathcal{T}}(x) - x \cdot k_\theta$.

Proposition 3.4. (1) For any $w \in \mathbb{R}^{V(\mathcal{T})}$, $cov_{\mathcal{T}}(w + x) - (w + x) \cdot k_\theta = con_{\mathcal{T}}(x) - x \cdot k_\theta$.
(2) If $\theta(q) > 0$ for all q , then

$$\lim_{x \rightarrow \infty, x \in \mathbb{R}^{E(\mathcal{T})}/\mathbb{R}^{V(\mathcal{T})}} [cov_{\mathcal{T}}(x) - x \cdot k] = +\infty.$$

In particular, $cov_{\mathcal{T}} - x \cdot k_\theta$ has a minimal point in $\mathbb{R}^{E(\mathcal{T})}$.

Proof. We can rewrite the function as

$$cov_{\mathcal{T}}(x) - x \cdot k_\theta = \sum_{\sigma \in T} \left[\sum_{q \subset \sigma} 2\Lambda(\alpha(q)) + (\alpha(q) - \theta(q)) \sum_{e \sim q} x(e) \right].$$

Therefore, statement (1) follows from that of covolume function for single tetrahedron (3.4). Part (2) follows from the condition that $\theta(q) > 0$ for all q , lemma 3.3 and (3.5). Since the function $\text{cov}_{\mathcal{T}}(x) - x \cdot k$ is convex on \mathbb{R}^E and is invariant under the linear action of \mathbb{R}^V , it is a convex function on the Euclidean space $\mathbb{R}^E/\mathbb{R}^V$ which tends to infinity as x tends to infinity. Therefore, $\text{cov}_{\mathcal{T}}(x) - x \cdot k$ has a minimal point in $\mathbb{R}^E/\mathbb{R}^V$ which implies that it has a minimal point in \mathbb{R}^E . \square

3.3 Fenchel dual of volume

The space of all cone angles $\mathcal{C}(\mathcal{T})$ defined by (3.3) is a compact convex polytope in \mathbb{R}^E . Indeed, it is the image of the compact convex polytope of all angle assignments in \mathbb{R}^{\square} under a linear map. Define $W : \mathbb{R}^E \rightarrow \mathbb{R}$ to be the function

$$W(k) = \begin{cases} 2 \min\{-\text{vol}(a) \mid a \in \mathcal{A}_k^*(M, \mathcal{T})\} & \text{if } k \in \mathcal{C}(\mathcal{T}), \\ +\infty & \text{if } k \notin \mathcal{C}(\mathcal{T}). \end{cases} \quad (3.6)$$

The function $W(k)$ encodes the volume optimization program. For instance, $W(2\pi, \dots, 2\pi)$ is the Casson-Rivin's program of finding complete hyperbolic metrics of finite volume.

Proposition 3.5. *The function $W : \mathcal{C}(\mathcal{T}) \rightarrow \mathbb{R}$ is convex and lower semi-continuous in the compact convex set $\mathcal{C}(\mathcal{T})$.*

This proposition follows from the lemma 3.6 below by taking $X = \{x \in \mathbb{R}_{\geq 0}^{\square} \mid \forall \sigma \in T, \sum_{q \subset \sigma} x(q) = \pi\}$, $f(x) = -2 \sum_{q \in \square} \Lambda(x(q)) = -2 \sum_{\sigma \in T} \sum_{q \subset \sigma} \Lambda(x(q))$ and $L : \mathbb{R}^{\square} \rightarrow \mathbb{R}^E$ to be $L(x)(e) = \sum_{q \sim e} x(q)$.

Lemma 3.6. *Suppose $X \subset \mathbb{R}^m$ is a compact convex set, $f : X \rightarrow \mathbb{R}$ is a continuous convex function and $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear. Then $g(y) = \min\{f(x) \mid x \in X, L(x) = y\}$ is convex and lower semi-continuous on $L(X)$.*

Proof. Take $y_1, y_2 \in L(X)$ and choose $x_1, x_2 \in X$ so that $L(x_i) = y_i$ and $g(y_i) = f(x_i)$ for $i = 1, 2$ and $t \in [0, 1]$. Then $L(tx_1 + (1-t)x_2) = ty_1 + (1-t)y_2$. Therefore, $g(ty_1 + (1-t)y_2) \leq f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) = tg(y_1) + (1-t)g(y_2)$, i.e., g is convex. To see lower semi continuity of g , suppose $y_n \in L(X)$ so that $\lim_n y_n = b \in L(X)$. By the compactness of X , after selecting a subsequence, we may assume that $y_n = L(x_n)$ so that $\lim_n x_n = a$ in X and $g(y_n) = f(x_n)$. Clearly due to continuity $\lim_n f(x_n) = f(a)$ and $L(a) = b$. This shows that $\lim_n g(y_n) = \lim_n f(x_n) = f(a) \geq g(b)$. \square

One of the two main technical results of the paper is the following,

Theorem 3.7. *The Fenchel dual of the C^1 smooth convex covolume $\text{cov} : \mathbb{R}^E \rightarrow \mathbb{R}$ is the lower semi-continuous convex function $W : \mathbb{R}^E \rightarrow (-\infty, \infty]$ defined by (3.6).*

Proof. By the Fenchel duality theorem for convex function, it suffices to show the dual $\text{cov}^*(y) = \sup\{x \cdot y - \text{cov}(x) \mid x \in \mathbb{R}^E\}$ of cov is the function W . Take $y \in \mathbb{R}^E$. We will show that if $y \notin \mathcal{C}(\mathcal{T})$, then $\text{cov}^*(y) = \infty$ and if $y \in \mathcal{C}(\mathcal{T})$, then $\text{cov}^*(y) = W(y)$.

Case 1. $y \notin \mathcal{C}(\mathcal{T})$. Suppose otherwise that $\text{cov}^*(y) < \infty$, i.e., there exists $C_1 > 0$ so that $x \cdot y - \text{cov}(x) \leq C_1$ for all $x \in \mathbb{R}^E$. Since $\text{vol}(x)$ is uniformly bounded for all x and $\text{cov}(x) - x \cdot k_x = 2\text{vol}(x)$, there exists $C_2 > 0$ so that for all $x \in \mathbb{R}^E$,

$$x \cdot y - x \cdot k_x \leq C_2. \quad (3.7)$$

where k_x is the cone angle of x considered as a generalized decorated metric.

Consider the linear map $L : \mathbb{R}^\square \rightarrow \mathbb{R}^E \times \mathbb{R}^T$ defined by

$$L(z)(e) = \sum_{q \sim e} z(q), \quad L(z)(\sigma) = \sum_{q \subset \sigma} z(q).$$

By the assumption that $y \notin \mathcal{C}(\mathcal{T})$, $(y, \pi, \dots, \pi) \notin L(\mathbb{R}_{\geq 0}^\square)$. Since $L(\mathbb{R}_{\geq 0}^\square)$ is a closed convex cone in $\mathbb{R}^E \times \mathbb{R}^T$, by the separation theorem applied to (y, π, \dots, π) and $L(\mathbb{R}_{\geq 0}^\square)$, there exists $h \in \mathbb{R}^E \times \mathbb{R}^T$ so that

$$\langle h, (y, \pi, \dots, \pi) \rangle = C_2 + 1,$$

and for all $t \in \mathbb{R}_{\geq 0}^\square$

$$\langle h, L(t) \rangle \leq 0$$

where $\langle u, v \rangle$ is the standard inner product in $\mathbb{R}^E \times \mathbb{R}^T$.

We can rewrite above two inequalities as

$$\sum_{e \in E} h(e)y(e) + \pi \sum_{\sigma \in T} h(\sigma) = C_2 + 1, \quad (3.8)$$

and,

$$\sum_{e \in E} [h(e) \sum_{q \sim e} t(q)] + \sum_{\sigma \in T} [h(\sigma) \sum_{q \subset \sigma} t(q)] \leq 0. \quad (3.9)$$

Equation (3.9) can be written as,

$$\sum_{q \in \square} t(q) \left(\sum_{e \sim q} h(e) + \sum_{q \subset \sigma} h(\sigma) \right) \leq 0.$$

Since it holds for all $t \in \mathbb{R}_{\geq 0}^\square$, hence for all $q \in \square$

$$\sum_{e \sim q} h(e) + \sum_{q \subset \sigma} h(\sigma) \leq 0. \quad (3.10)$$

Now since (3.7) holds for all $x \in \mathbb{R}^E$, it holds for the projection of $h \in \mathbb{R}^E \times \mathbb{R}^T$ to \mathbb{R}^E . Taking this projection as x in (3.7), letting the dihedral angle of x be $\alpha = \alpha_x$ and using (3.8) and (3.9), we obtain

$$\begin{aligned} C_2 &\geq \sum_{e \in E} h(e)y(e) - \sum_{e \in E} h(e) \left(\sum_{q \sim e} \alpha(q) \right) \\ &= C_2 + 1 - \pi \sum_{\sigma \in T} h(\sigma) - \sum_{e \in E} \left(\sum_{q \sim e} \alpha(q) \right) h(e) \\ &= C_2 + 1 - \sum_{\sigma \in T} h(\sigma) \left(\sum_{q \subset \sigma} \alpha(q) \right) - \sum_{e \in E} \left(\sum_{q \sim e} \alpha(q) \right) h(e) \\ &= C_2 + 1 - \sum_{q \in \square} \alpha(q) \left(\sum_{q \subset \sigma} h(\sigma) + \sum_{e \sim q} h(e) \right) \\ &\geq C_2 + 1. \end{aligned}$$

Here we have used (3.8) and (3.10) in steps 2 and 4 in the derivation above. This is a contradiction. Therefore, $\text{cov}^*(y) = \infty$.

Case 2. $y \in \mathcal{C}(\mathcal{T})$, say $y = k_\theta$ for an angle assignment $\theta \in \mathbb{R}_{\geq 0}^\square$ where $y(e) = \sum_{q \sim e} \theta(q)$. For this choice of θ , and for any $x \in \mathbb{R}^E$ with dihedral angles $\alpha(q) = \alpha_x(q)$, we can write

$$\begin{aligned} x \cdot y - \text{cov}(x) &= \sum_e x(e)y(e) - 2\text{vol}(x) - x \cdot k_x \\ &= - \sum_{\sigma \in T} \left[\sum_{q \subset \sigma} (2\Lambda(\alpha(q)) + \sum_{e < \sigma} x(e) \sum_{q \sim e} (\alpha(q) - \theta(q))) \right]. \end{aligned}$$

Lemma 3.8. *If $\theta_1, \theta_2, \theta_3 \geq 0$ and $\theta_1 + \theta_2 + \theta_3 = \pi$, then for any $x_1, x_2, x_3 \in \mathbb{R}$ so that $\alpha_1, \alpha_2, \alpha_3$ are inner angles of the generalized Euclidean triangle of edge lengths $e^{x_1}, e^{x_2}, e^{x_3}$, we have*

$$\sum_{i=1}^3 \Lambda(\alpha_i) + \sum_{i=1}^3 x_i(\alpha_i - \theta_i) \geq \sum_{i=1}^3 \Lambda(\theta_i).$$

Proof. The lemma follows from proposition 2.6 on convexity and C^1 -smoothness of the covolume function $\text{cov} : \mathbb{R}^3 \rightarrow \mathbb{R}$ (for single tetrahedron). For any C^1 -smooth convex function F on \mathbb{R}^n , we have $F(x) - DF(l)(x - l) \geq F(l)$. Now using the fact that $\frac{\partial \text{cov}}{\partial x_i} = \alpha_i$ and take $l \in \mathbb{R}^3$ so that the angles of the generalized Euclidean triangle of edge lengths $e^{l_1}, e^{l_2}, e^{l_3}$ are θ_i 's, the result follows. \square

Summing up the inequalities in lemma 3.8 over all tetrahedra in T , we obtain

$$2\text{vol}(x) + x \cdot k_x - x \cdot k_\theta \geq 2\text{vol}(\theta).$$

This implies, for any angle assignment $\theta \in \mathcal{A}_y^*(M, \mathcal{T})$, we have

$$x \cdot y - \text{cov}(x) \leq -2\text{vol}(\theta),$$

and hence

$$x \cdot y - \text{cov}(x) \leq W(y). \quad (3.11)$$

Taking the supremum in x of the inequality above, we obtain

$$\text{cov}^*(y) \leq W(y).$$

We claim that the equality holds. There are two steps involved in the proof. In the first step, we assume that $y = k_\theta$ where $\theta \in \mathcal{A}_y(M, \mathcal{T})$. In this case, by proposition 3.4, the function $\text{cov}(x) - x \cdot y$ of variable x has a minimal point x^* in \mathbb{R}^E . Let θ^* be the dihedral angles of x^* . Then by $\frac{\partial \text{cov}(x)}{\partial x_e} = k_x(e)$, we obtain $k_{x^*} = y$ (due to $\frac{\partial(\text{cov}(x) - x \cdot y)}{\partial x_e} = k_x(e) - y(e) = 0$ for $x = x^*$). Therefore, at this point x^* , $x^* \cdot y - \text{cov}(x^*) = -2\text{vol}(\theta^*) \geq W(y)$. This shows $\text{cov}^*(y) \geq W(y)$. Combining with (3.11), we conclude $\text{cov}^*(y) = W(y)$.

Next, both $\text{cov}^*(y)$ and $W(y)$ are convex and semi-continuous on the closed convex set $\mathcal{C}(\mathcal{T})$ so that they coincide in the subset $\{y \mid \mathcal{A}_y(M, \mathcal{T}) \neq \emptyset\}$ of $\mathcal{C}(\mathcal{T})$ which contains the relative interior of $\mathcal{C}(\mathcal{T})$. Therefore, $\text{cov}^*(y) = W(y)$ on $\mathcal{C}(\mathcal{T})$ is a consequence of the lemma below. \square

Lemma 3.9 ([32], corollary 7.3.4). *Suppose $X \subset \mathbb{R}^n$ is a closed convex set and $f, g : X \rightarrow \mathbb{R}$ are convex semi-continuous functions. If f and g coincide on the relative interior of X , then $f = g$.*

3.4 Proofs of main theorems

We begin with the following,

Proposition 3.10. *Suppose $k \in \mathbb{R}^E$ so that $\mathcal{A}_k(M, \mathcal{T}) \neq \emptyset$. Then there is a unique maximum point of the volume function $\text{vol} : \mathcal{A}_k^*(M, \mathcal{T}) \rightarrow \mathbb{R}$.*

Proof. The maximum point exists since the vol is continuous on the compact set $\mathcal{A}_k^*(M, \mathcal{T})$. Suppose otherwise that α, α' are two distinct maximum points of volume in $\mathcal{A}_k^*(M, \mathcal{T})$. Then due to convexity, all points in the line segment $t\alpha + (1-t)\alpha'$ for $t \in [0, 1]$ are maximum point. On the other hand, the work of Rivin [33] shows that, due to $\mathcal{A}_k(M, \mathcal{T}) \neq \emptyset$, for any maximum volume point $\beta \in \mathcal{A}_k^*(M, \mathcal{T})$ if $\beta(q) = 0$ for some $q \subset \sigma$, then for the other two quads $q', q'' \subset \sigma$, $\{\beta(q'), \beta(q'')\} = \{0, \pi\}$. This shows if $\alpha(q) = \pi$, then $\alpha'(q) = \pi$. For otherwise, the values of the maximum point $\frac{1}{2}(\alpha + \alpha')$ at the three quads $q, q', q'' \subset \sigma$ would be $0, \mu, \pi - \mu$ for some $\mu \in (0, \pi)$. Now due to $\mathcal{A}_k(M, \mathcal{T}) \neq \emptyset$, $\text{vol}(\alpha) > 0$. Therefore, there is a tetrahedron $\sigma \in T$ so that $\alpha(q), \alpha'(q) > 0$ for all $q \subset \sigma$. Hence the function $g(t) = \text{vol}(t\alpha + (1-t)\alpha') = \sum_{\sigma \in T} \sum_{q \subset \sigma} \Lambda(t\alpha(q) + (1-t)\alpha'(q))$ is a sum of concave functions so that one of it is strictly concave in t . This implies $g(t)$ is not a constant function which contradicts that $g(t)$'s are the maximum value $\text{vol}(\alpha)$. \square

3.4.1 A proof of theorem 3.1

To prove part (1) of theorem 3.1, take any $\theta \in \mathcal{A}_k^*(M, \mathcal{T})$ whose cone angle is k . By definition $W(k) \leq -2\text{vol}(\theta)$. But by the duality theorem 3.7, $W(k) = \text{cov}^*(k) = \sup\{y \cdot k - \text{cov}(y) \mid y \in \mathbb{R}^E\} \geq x \cdot k - \text{cov}(x) = -2\text{vol}(\alpha)$. Thus $\text{vol}(\alpha) \geq \text{vol}(\theta)$, i.e., α is a maximum volume point.

To prove part (2) of theorem 3.1, since $\mathcal{A}_k(M, \mathcal{T}) \neq \emptyset$, by proposition 3.4, the function $\text{cov}(x) - x \cdot k$ has a minimal point $l \in \mathbb{R}^E$. Let β be the dihedral angle of l . Then by $\nabla(\text{cov}(x) - x \cdot k) = 0$ at $x = l$, we see that the cone angle of β is k , i.e., $\beta \in \mathcal{A}_k^*(M, \mathcal{T})$. By part (1), β is a maximum volume point in $\mathcal{A}_k^*(M, \mathcal{T})$. Therefore, both β and α are maximum volume points in $\mathcal{A}_k^*(M, \mathcal{T})$. By proposition 3.10, $\alpha = \beta$. Thus the result follows.

3.4.2 A proof of theorem 3.2

To prove theorem 3.2, suppose that $x, y \in \mathbb{R}^E$ are two decorated ideal hyperbolic polyhedral metrics on (M, \mathcal{T}) so that their cone angles k_x and k_y are the same. We will show that $y = w + x$ for some $w \in \mathbb{R}^{V(\mathcal{T})}$, i.e., x, y differ by a change of decoration. Let $k = k_x$. By the assumption that x is a decorated ideal hyperbolic polyhedral metric, $\mathcal{A}_k(M, \mathcal{T}) \neq \emptyset$. By theorem 3.1, the dihedral angles α_x and α_y of x and y are the maximum volume points in $\mathcal{A}_k^*(M, \mathcal{T})$. On the other hand, proposition 3.10 shows the maximum volume point is unique. Therefore $\alpha_x = \alpha_y$. This implies that the underlying hyperbolic metrics for x and y are isometric. Therefore, x and y differ by a change of decoration, i.e., $x = w + y$, for some $w \in \mathbb{R}^V$.

Note that we have proved a slightly stronger statement that we only need to assume x is decorated hyperbolic metric and y is a generalized decorated metric of the same cone angle.

4 Hyper-ideal tetrahedra

4.1 Preliminaries

We recall some of the basic results on hyper-ideal tetrahedra in this subsection. Following [2] and [14], a *hyper-ideal tetrahedron* σ in \mathbb{H}^3 is a compact convex polyhedron that is diffeomorphic to a truncated tetrahedron in \mathbb{E}^3 with four hexagonal faces right-angled hyperbolic hexagons (see Figure 1 (a)). The

four triangular faces isometric to hyperbolic triangles are called *vertex triangles*. An *edge* in a hyper-ideal tetrahedron is the intersection of two hexagonal faces, and a *vertex edge* is the intersection of a hexagonal face and a vertex triangle. The *dihedral angle* at an edge is the angle between the two hexagonal faces adjacent to it. The dihedral angle between a hexagonal face and a vertex triangle is always $\pi/2$.

Let Δ_i , $i = 1, 2, 3, 4$, be the four vertex triangles of σ . We use e_{ij} to denote the edge joining Δ_i to Δ_j , and use H_{ijk} to denote the hexagonal face adjacent to e_{ij} , e_{jk} and e_{ik} . (See Figure 1 (a).) The length of e_{ij} is denoted by l_{ij} and the dihedral angle at e_{ij} is denoted by a_{ij} . The length of the vertex edge $\Delta_i \cap H_{ijk}$ is denoted by x_{jk}^i . As a convention, we always assume $l_{ij} = l_{ji}$ and $a_{ij} = a_{ji}$.

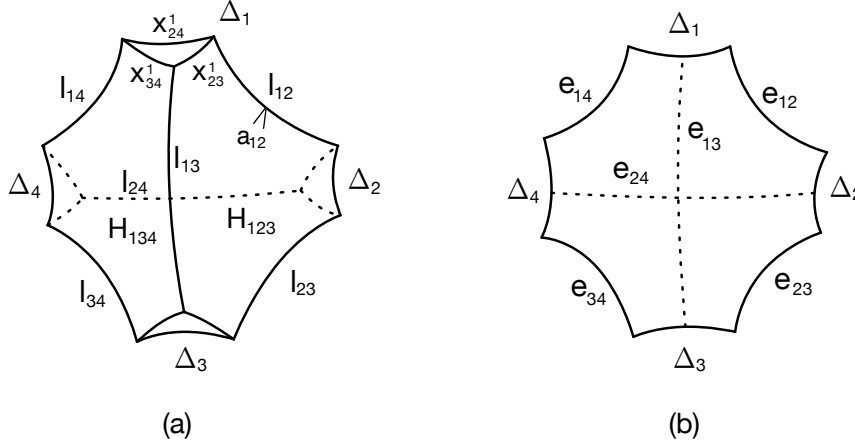


Figure 1: Hyper-ideal and flat hyper-ideal tetrahedra.

Proposition 4.1. ([2], [14]) Suppose σ is a hyper-ideal tetrahedron in \mathbb{H}^3 .

- (a) The isometry class of σ is determined by its dihedral angle vector $(a_{12}, \dots, a_{34}) \in \mathbb{R}^6$ which satisfies the condition that $a_{ij} > 0$, and $\sum_{j \neq i} a_{ij} < \pi$ for each fixed i .
- (b) Conversely, given $(a_{12}, \dots, a_{34}) \in \mathbb{R}_{>0}^6$ so that $\sum_{j \neq i} a_{ij} < \pi$ for each i , where $a_{ij} = a_{ji}$, there exists a hyper-ideal tetrahedron having a_{ij} as its dihedral angle at the ij -th edge.
- (c) The isometry class of σ is determined by its edge length vector $(l_{12}, \dots, l_{34}) \in \mathbb{R}_{>0}^6$.

Thus, the space of isometry classes of hyper-ideal tetrahedra parametrized by dihedral angles is the open convex polytope

$$\mathcal{B} = \left\{ (a_{12}, \dots, a_{34}) \in \mathbb{R}_{>0}^6 \mid \sum_{j \neq i} a_{ij} < \pi \text{ for each } i, \text{ where } a_{ij} = a_{ji} \right\}. \quad (4.1)$$

Let $\text{vol}: \mathcal{B} \rightarrow \mathbb{R}$ be the hyperbolic volume of the hyper-ideal tetrahedra considered as a function in the dihedral angles. Then the Schlaefli formula says

$$\frac{\partial \text{vol}}{\partial a_{ij}} = -\frac{l_{ij}}{2}.$$

See [1] for the Schlaefli formula in a more general setting.

Proposition 4.2. The volume function has the following properties.

- (a) ([36]) The volume function $\text{vol}: \mathcal{B} \rightarrow \mathbb{R}$ is smooth and has positive definite Hessian matrix at each point in \mathcal{B} .
- (b) ([35]) The function vol can be extended continuously to the compact closure $\overline{\mathcal{B}}$ of \mathcal{B} in \mathbb{R}^6 , where

$$\overline{\mathcal{B}} = \left\{ (a_{12}, \dots, a_{34}) \in \mathbb{R}_{\geq 0}^6 \mid \sum_{j \neq i} a_{ij} \leq \pi \text{ for each } i, \text{ where } a_{ij} = a_{ji} \right\}.$$

Let \mathcal{L} be the set of vectors (l_{12}, \dots, l_{34}) such that there exists a hyper-ideal tetrahedron having l_{ij} as the length of the ij -th edge. It follows from Proposition 4.1 that \mathcal{L} is a simply connected open subset of \mathbb{R}^6 . The volume function can be considered as defined on \mathcal{L} . The Legendre transform of $\text{vol}: \mathcal{B} \rightarrow \mathbb{R}$, to be called the *co-volume* function, is $\text{cov}: \mathcal{L} \rightarrow \mathbb{R}$ given by

$$\text{cov}(l) = 2\text{vol}(l) + \sum_{i < j} a_{ij} l_{ij}. \quad (4.2)$$

It is known (see [21]) that cov has a positive definite Hessian matrix at each $l \in \mathcal{L}$ and hence cov is locally strictly convex. However, the open subset $\mathcal{L} \subset \mathbb{R}_{>0}^6$ is not convex. One of the main technical results in this paper is that cov can be extended to a C^1 -smooth and convex function on \mathbb{R}^6 by studying the flat hyper-ideal tetrahedra.

Recall that a *flat hyper-ideal tetrahedron* is defined as follows. Take a right-angled hyperbolic octagon Q with eight edges cyclically labelled as $\Delta_1, e_{12}, \Delta_2, e_{23}, \Delta_3, e_{34}, \Delta_4, e_{41}$. Let e_{13} (and e_{24}) be the shortest geodesic arc in Q joining Δ_1 to Δ_3 (and Δ_2 and Δ_4). We call $(Q, \{e_{ij}\})$ a *flat hyper-ideal tetrahedron* with six edges e_{ij} . See Figure 1 (b). The dihedral angles at e_{13} and e_{24} are π and are 0 at all other edges. The volume of a flat hyper-ideal tetrahedron is defined to be zero.

4.2 Generalized hyper-ideal tetrahedra, dihedral angles and volume

In this subsection, we investigate the space of hyper-ideal tetrahedra parametrized by the edge lengths and their degenerations. One of the goals is to extend the locally convex function cov to a convex function defined on $\mathbb{R}_{>0}^6$. To this end, let us define a *generalized hyper-ideal tetrahedron* to be a topological truncated tetrahedron so that each edge is assigned a positive number, called the edge length. We will define the *dihedral angles* and *volume* and *covolume* of generalized hyper-ideal tetrahedra in this section.

Suppose σ is a generalized hyper-ideal tetrahedron with edges e_{ij} joining the i -th and the j -th vertices and $l_{ij} = l_{ji}$ is the edge length of e_{ij} . To define the dihedral angle a_{ij} at e_{ij} , we need the following compatibility property, which is a special case of Proposition 3.1 of [22].

Lemma 4.3. For $(l_{12}, \dots, l_{34}) \in \mathbb{R}_{>0}^6$ and $\{i, j, k, h\} = \{1, 2, 3, 4\}$, let $l_{ji} = l_{ij}$ for $i \neq j$ and let

$$x_{jk}^i = \cosh^{-1} \left(\frac{\cosh l_{ij} \cosh l_{ik} + \cosh l_{jk}}{\sinh l_{ij} \sinh l_{ik}} \right) \quad (4.3)$$

and

$$\phi_{kh}^i = \frac{\cosh x_{jk}^i \cosh x_{jh}^i - \cosh x_{kh}^i}{\sinh x_{jk}^i \sinh x_{jh}^i}. \quad (4.4)$$

Then $\phi_{kh}^i = \phi_{kh}^j$.

Proof. Let $c_{ij} = \cosh l_{ij}$, $s_{ij} = \sinh l_{ij}$, $c_{ij}^k = \cosh x_{ij}^k$ and $s_{ij}^k = \sinh x_{ij}^k$ for $\{i, j, k\} \subset \{1, \dots, 4\}$. By definition, we have

$$\begin{aligned}\phi_{kh}^i &= \frac{1}{s_{jk}^i s_{jh}^i} \left(\frac{c_{ij} c_{ik} + c_{jk}}{s_{ij} s_{ik}} \frac{c_{ij} c_{ih} + c_{jh}}{s_{ij} s_{ih}} - \frac{c_{ik} c_{ih} + c_{kh}}{s_{ik} s_{ih}} \right) \\ &= \frac{c_{ik} c_{ih} + c_{jk} c_{jh} + c_{ij} c_{ik} c_{jh} + c_{ij} c_{ih} c_{jk} - s_{ij}^2 c_{kh}}{s_{jk}^i s_{jh}^i s_{ij}^2 s_{ik} s_{ih}},\end{aligned}\tag{4.5}$$

and similarly

$$\phi_{kh}^j = \frac{c_{jk} c_{jh} + c_{ik} c_{ih} + c_{ij} c_{ih} c_{jk} + c_{ij} c_{ik} c_{jh} - s_{ij}^2 c_{kh}}{s_{ik}^j s_{ih}^j s_{ij}^2 s_{jk} s_{jh}}.\tag{4.6}$$

To see $\phi_{kh}^i = \phi_{kh}^j$, it suffices to show that the two denominators in (4.5) and (4.6) are the same. To this end, we have

$$\begin{aligned}(s_{jk}^i s_{ij} s_{ik})^2 &= ((c_{jk}^i)^2 - 1) s_{ij}^2 s_{ik}^2 \\ &= \left(\left(\frac{c_{ij} c_{ik} + c_{jk}}{s_{ij} s_{ik}} \right)^2 - 1 \right) s_{ij}^2 s_{ik}^2 \\ &= 2c_{ij} c_{ik} c_{jk} + c_{ij}^2 + c_{ik}^2 + c_{jk}^2 - 1.\end{aligned}$$

Therefore,

$$\phi_{kh}^i(l) = \phi_{kh}^j(l) = \frac{c_{ik} c_{ih} + c_{jk} c_{jh} + c_{ij} c_{ik} c_{jh} + c_{ij} c_{ih} c_{jk} - s_{ij}^2 c_{kh}}{\sqrt{2c_{ij} c_{ik} c_{jk} + c_{ij}^2 + c_{ik}^2 + c_{jk}^2 - 1} \sqrt{2c_{ij} c_{ih} c_{jh} + c_{ij}^2 + c_{ih}^2 + c_{jh}^2 - 1}}.\tag{4.7}$$

□

Note that if σ is a hyper-ideal tetrahedron of edge lengths l_{ij} , then by the cosine law, x_{jk}^i and $\arccos(\phi_{kh}^i)$ in lemma 4.3 are the lengths of the vertex edge $\Delta_i \cap H_{ijk}$ and the dihedral angle at e_{kh} . In particular, the conclusion of the lemma is obvious for σ . For a generalized hyper-ideal tetrahedron $(\sigma, \{l_{ij}\})$, using lemma 4.3 we call x_{jk}^i the *length* of the vertex edge and define the function $\phi_{ij}: \mathbb{R}_{>0}^6 \rightarrow \mathbb{R}$ by $\phi_{ij}(l) = \phi_{kh}^i(l)$. Then we have

Proposition 4.4. *The space of all hyper-ideal tetrahedra parametrized by the edge lengths is*

$$\mathcal{L} = \{l \in \mathbb{R}_{>0}^6 \mid \phi_{ij}(l) \in (-1, 1) \text{ for all } \{i, j\} \subset \{1, 2, 3, 4\}, i \neq j\}.$$

Proof. If l is the edge length vector of a hyperbolic tetrahedra, then each of the dihedral angles $a_{ij}(l) \in (0, \pi)$. By the cosine law for the hyperbolic triangle Δ_k and right-angled hexagon H_{ijk} , we have $\phi_{ij}(l) = \cos a_{ij}(l) \in (-1, 1)$. This shows $\phi_{ij} \in (-1, 1)$. Conversely, for each $l \in \mathbb{R}_{>0}^6$ with $\phi_{ij}(l) \in (-1, 1)$ for all $\{i, j\} \subset \{1, \dots, 4\}$, by (4.4), x_{jk}^i , x_{jh}^i and x_{kh}^i satisfy the triangular inequality. Then there exists a unique hyperbolic triangle Δ_i having them as edge lengths. Taking $a_{ij} = \cos^{-1}(\phi_{ij}) \in (0, \pi)$, by Lemma 4.3, we see that a_{ij} , a_{ik} and a_{ih} are the angles of Δ_i . Hence they satisfy $a_{ij} + a_{ik} + a_{ih} < \pi$. By Proposition 4.1, there is a unique hyper-ideal tetrahedron σ with dihedral angles $\{a_{ij}\}$. Applying the Cosine Law to the vertex triangles $\{\Delta_i\}$ and hexagons $\{H_{ijk}\}$, we see that l is the edge lengths of σ . □

Proposition 4.5. *Let $\partial\mathcal{L}$ be the frontier of \mathcal{L} in $\mathbb{R}_{>0}^6$. Then $\partial\mathcal{L} = X_1 \sqcup X_2 \sqcup X_3$, where each X_i , $i = 1, 2, 3$, is a real analytic codimension-1 submanifold of $\mathbb{R}_{>0}^6$. The complement $\mathbb{R}_{>0}^6 \setminus \mathcal{L}$ is a disjoint union of three manifolds Ω_i with boundary so that $\Omega_i \cap \partial\mathcal{L} = X_i$, $i = 1, 2, 3$.*

For $\{i, j\} \subset \{1, \dots, 4\}$, let $\Omega_{ij}^\pm = \{l \in \mathbb{R}_{>0}^6 \mid \pm \phi_{ij}(l) \geq 1\}$ and $X_{ij}^\pm = \{l \in \mathbb{R}_{>0}^6 \mid \phi_{ij}(l) = \pm 1\}$. Then by Proposition 4.4, we have

$$\mathbb{R}_{>0}^6 \setminus \mathcal{L} = \bigcup_{i \neq j} (\Omega_{ij}^+ \cup \Omega_{ij}^-).$$

Lemma 4.6. For $\{i, j, k, h\} = \{1, 2, 3, 4\}$, we have

- (1) $\Omega_{ij}^- \cap \Omega_{ik}^- = \emptyset$.
- (2) $\Omega_{ij}^- = \Omega_{kh}^-$ and $\Omega_{ij}^+ = \Omega_{ik}^- \cup \Omega_{ih}^-$.
- (3) $X_{ij}^- = X_{kh}^-$ and $X_{ij}^+ = X_{ik}^- \cup X_{ih}^-$.

Proof. For each $l \in \Omega_{ij}^-$, we have $\phi_{kh}^i(l) \leq -1$ and $x_{jk}^i + x_{jh}^i \leq x_{kh}^i$, which implies $\phi_{jk}^i(l) \geq 1$ and $\phi_{jh}^i(l) \geq 1$. Therefore, (1) holds and $\Omega_{ij}^- \subset \Omega_{ik}^+ \cap \Omega_{ih}^+$. On the other hand, for each $l \in \Omega_{ik}^+ \cap \Omega_{ih}^+$, we have $\phi_{jk}^i(l) \geq 1$ and $\phi_{jh}^i(l) \geq 1$, which implies that $x_{jh}^i \leq |x_{jk}^i - x_{kh}^i|$ and $x_{jk}^i \leq |x_{jh}^i - x_{kh}^i|$. As a consequence, $x_{jk}^i + x_{jh}^i \leq x_{kh}^i$ and $\phi_{kh}^i(l) \leq -1$. Therefore, we have $\Omega_{ik}^+ \cap \Omega_{ih}^+ \subset \Omega_{ij}^-$, hence $\Omega_{ij}^- = \Omega_{ik}^+ \cap \Omega_{ih}^+$. By symmetry, we have $\Omega_{ij}^- = \Omega_{jk}^+ \cap \Omega_{jh}^+$, from which we see $\Omega_{ij}^- = \Omega_{ik}^+ \cap \Omega_{ih}^+ \cap \Omega_{jk}^+ \cap \Omega_{jh}^+ = \Omega_{kh}^-$. Now for $l \in \Omega_{ij}^+$, we have $\phi_{kh}^i(l) \geq 1$, which implies that $x_{kh}^i \leq |x_{jk}^i - x_{jh}^i|$. If $x_{kh}^i \leq x_{jk}^i - x_{jh}^i$, then $\phi_{jh}^i(l) \leq -1$ and $l \in \Omega_{ik}^-$. If $x_{kh}^i \leq x_{jh}^i - x_{jk}^i$, then $\phi_{jk}^i(l) \leq -1$ and $l \in \Omega_{ih}^-$. Therefore, we have $\Omega_{ij}^+ \subset \Omega_{ik}^- \cup \Omega_{ih}^-$. On the other hand, since $\Omega_{ik}^- \subset \Omega_{ij}^+$ and $\Omega_{ih}^- \subset \Omega_{ij}^+$, we have $\Omega_{ik}^- \cup \Omega_{ih}^- \subset \Omega_{ij}^+$, from which (2) follows. (3) follows from the same argument with the inequalities replaced by equalities. \square

Proof of Proposition 4.5. Let $b_h = \sqrt{2c_{ij}c_{ik}c_{jk} + c_{ij}^2 + c_{ik}^2 + c_{jk}^2} - 1$, $c_{ij} = \cosh l_{ij}$ and $s_{ij} = \sinh l_{ij}$ for each $l = (l_{12}, \dots, l_{34}) \in \mathbb{R}_{>0}^6$ and $l_{ij} = l_{ji}$. We have

$$\frac{\partial \phi_{ij}}{\partial l_{kh}} = -\frac{s_{ij}^2 s_{kh}}{b_k b_h} \neq 0,$$

which implies $\nabla \phi_{ij} \neq 0$. Therefore, -1 is a regular values of ϕ_{ij} . By the Implicit Function Theorem, $X_{ij}^- = \phi_{ij}^{-1}(-1)$ is a smooth codimension-1 submanifold of $\mathbb{R}_{>0}^6$. Since each ϕ_{ij} is real analytic in $\mathbb{R}_{>0}^6$, the submanifold X_{ij}^- is real analytic. Let $\Omega_1 = \Omega_{12}^-$, $\Omega_2 = \Omega_{13}^-$ and $\Omega_3 = \Omega_{14}^-$, and similarly let $X_1 = X_{12}^-$, $X_2 = X_{13}^-$ and $X_3 = X_{14}^-$. As a consequence of Lemma 4.6, we have

$$\mathbb{R}_{>0}^6 \setminus \mathcal{L} = \Omega_1 \sqcup \Omega_2 \sqcup \Omega_3$$

is a disjoint union of three 6-dimensional submanifolds with boundary. By Lemma 4.6, $X_i = \partial \Omega_i$ and $\partial \mathcal{L} \subset \sqcup_{i=1}^3 \partial \Omega_i = \sqcup_{i=1}^3 X_i$. We claim that $X_i \subset \partial \mathcal{L}$. Indeed, for each $l = (l_{12}, \dots, l_{34}) \in X_1$, say, we construct a sequence $\{l^{(n)}\} \subset \mathcal{L}$ convergent to l as follows. We let $\epsilon_n \rightarrow 0^+$, and define $l^{(n)} = (l_{12} - \epsilon_n, l_{13}, \dots, l_{34}) \rightarrow l$. Then for n large enough, by (4.3) $x_{jk}^{i(n)}$, $x_{jh}^{i(n)}$ and $x_{kh}^{i(n)}$ satisfy the triangular inequalities, and by (4.4) each $\phi_{ij}(l^{(n)}) \in (-1, 1)$. By Proposition 4.4, $l^{(n)} \in \mathcal{L}$ for n large enough. Therefore, we have $\partial \mathcal{L} = \sqcup_{i=1}^3 X_i$, which completes the proof. \square

Lemma 4.7. The function ϕ_{ij} extends continuously to $\mathbb{R}_{\geq 0}^6$, and $\phi_{ij}(l) = 1$ when $l_{ij} = 0$.

Proof. Since the denominator of ϕ_{ij} is never equal to 0, the function continuously extends $\mathbb{R}_{\geq 0}^6$. Furthermore a direct calculation show if $l_{ij} = 0$, i.e., $c_{ij} = 1$, then $\phi_{ij}(l) = 1$. Indeed, both numerator and denominator in (4.7) are $(c_{ik} + c_{jk})(c_{ih} + c_{jh})$. \square

Proposition 4.8. *For each subset S of the edges of a tetrahedron, let*

$$\mathcal{D}_S = \{l \in \mathbb{R}_{\geq 0}^6 \mid l(e) > 0 \text{ for } e \in S \text{ and } l(e) = 0 \text{ for } e \notin S\},$$

and let $\overline{X_i}$, $i = 1, 2, 3$, be the closure of X_i in $\mathbb{R}_{\geq 0}^6$. If $\mathcal{D}_S \cap \overline{X_i} \neq \emptyset$, then $X_i^S \doteq \mathcal{D}_S \cap \overline{X_i}$ is a real analytic codimension-1 submanifold of \mathcal{D}_S .

Proof. Let $\overline{X_{ij}^-}$ be the closure of X_{ij}^- in $\mathbb{R}_{\geq 0}^6$. We first observe that if $e_{ij} \notin S$, then $\mathcal{D}_S \cap \overline{X_{ij}^-} = \emptyset$. Indeed, by Lemma 4.7, if $l \in \overline{X_{ij}^-}$, then $\phi_{ij}(l) = -1$, and if $l \in \mathbb{R}^S$, then $\phi_{ij}(l) = 1$. Therefore, if $l \in \mathcal{D}_S \cap \overline{X_{ij}^-}$, then $e_{ij} \in S$ and hence $s_{ij} \neq 0$. By Lemma 4.6, $X_{ij}^- = X_{kh}^-$, which implies $e_{kh} \in S$, hence $s_{kh} \neq 0$. Letting c_{ij} , s_{ij} and b_i be as before, we have

$$\frac{\partial \phi_{ij}}{\partial l_{kh}} = -\frac{s_{ij}^2 s_{kh}}{b_k b_h} \neq 0,$$

which implies that the projection of $\nabla \phi_{ij}$ to the tangent space of \mathcal{D}_S at $l \in \mathcal{D}_S \cap \overline{X_{ij}^-}$ is non-vanishing, i.e., \mathcal{D}_S and $\overline{X_{ij}^-}$ transversely intersect. By the Implicit Function Theorem, the intersection $\mathcal{D}_S \cap \overline{X_{ij}^-} = \mathcal{D}_S \cap \phi_{ij}^{-1}(-1)$ is a smooth codimension-1 submanifold of \mathcal{D}_S . Since each ϕ_{ij} is real analytic, the submanifold is real analytic in \mathcal{D}_S . \square

Let $\overline{\mathcal{L}}$ be the closure of \mathcal{L} in $\mathbb{R}_{\geq 0}^6$ and let $\mathcal{L}_S = \mathcal{D}_S \cap \overline{\mathcal{L}}$. For $i \neq j$, by defining $a_{ij}|_{\Omega_{ij}^+} = 0$ and $a_{ij}|_{\Omega_{ij}^-} = \pi$, we have

Corollary 4.9. *The dihedral angle function $a_{ij}: \mathcal{L} \rightarrow \mathbb{R}$ can be extended continuously to $\mathbb{R}_{\geq 0}^6$ so that its extension, still denoted by $a_{ij}: \mathbb{R}_{\geq 0}^6 \rightarrow \mathbb{R}$, is a constant on each component of $\mathcal{D}_S \setminus \mathcal{L}_S$ for each subset S of the edges.*

We call a_{ij} in corollary 4.9 the *dihedral angle* of the generalized hyper-ideal tetrahedron $(\sigma, \{l_{rs}\})$.

4.3 Covolume of generalized hyper-ideal tetrahedron

The locally convex function covolume function $cov: \mathcal{L} \rightarrow \mathbb{R}$ defined by (4.2) satisfies the Schläfli identity that

$$\frac{\partial cov}{\partial l_{ij}} = a_{ij}$$

for $i \neq j$, where $a_{ij}: \mathcal{L} \rightarrow \mathbb{R}$ is the dihedral angle function at the ij -th edge. In particular, the differential 1-form $\omega = \sum_{i < j} a_{ij} dl_{ij} = dcov$ is closed in \mathcal{L} , and we can recover covolume cov by the integration $cov(l) = \int^l \omega$.

For each $l = (l_{12}, \dots, l_{34}) \in \mathbb{R}^6$, we let $l^+ = (l_{12}^+, \dots, l_{34}^+) \in \mathbb{R}_{\geq 0}^6$ where $l_{ij}^+ = \max\{0, l_{ij}\}$. By Corollary 4.9, we can extend the function $a_{ij}: \mathcal{L} \rightarrow \mathbb{R}$ to a continuous function $a_{ij}: \mathbb{R}^6 \rightarrow \mathbb{R}$ by

$$a_{ij}(l) = a_{ij}(l^+),$$

and call $a(l) = (a_{12}(l), \dots, a_{34}(l))$ the *dihedral angle vector* of $l \in \mathbb{R}^6$. We define a new continuous 1-form μ on \mathbb{R}^6 by

$$\mu(l) = \sum_{i \neq j} a_{ij}(l) dl_{ij}.$$

Proposition 4.10. *The continuous differential 1-form $\mu = \sum_{i \neq j} a_{ij}(l) dl_{ij}$ is closed in \mathbb{R}^6 , i.e., for any Euclidean triangle Δ in \mathbb{R}^6 , $\int_{\partial \Delta} \mu = 0$.*

Proof. We prove it in two steps. In the first step, we prove that μ is closed in $\mathbb{R}_{\geq 0}^6$. Next, we show μ is closed in \mathbb{R}^6 .

By Corollary 4.9, the differential 1-form $\mu = \sum_{ij} a_{ij}(l) dl_{ij}$ is continuous in \mathbb{R}^6 . The restriction $\mu|_{\mathcal{L}} = \sum_{ij} a_{ij} dl_{ij} = dcov$ is closed. By corollary 4.9, a_{ij} is a constant in each connected component of $\mathbb{R}_{>0}^6 \setminus \mathcal{L}$. Proposition 4.5 shows that the subset \mathcal{L} in $\mathbb{R}_{>0}^6$ is open and bounded by a smooth codimension-1 submanifold. Now we use the following lemma.

Lemma 4.11. (Propositions 2.4 and 2.5, [23]) Suppose $U \subset \mathbb{R}^N$ is an open set and $\lambda = \sum_i \alpha_i(x) dx_i$ is a continuous 1-form on U .

(1) If $A \subset U$ is an open subset bounded by a smooth codimension-1 submanifold of U , and $\lambda|_A$ and $\lambda|_{U \setminus \bar{A}}$ are closed, then μ is closed in U .

(2) If U is simply connected, then $F(x) = \int^x \lambda$ is a C^1 -smooth function such that

$$\frac{\partial F}{\partial x_i} = \alpha_i.$$

(3) If U is convex and $A \subset U$ is an open subset of U bounded by a codimension-1 real analytic submanifold of U so that $F|_U$ and $F|_{U \setminus \bar{A}}$ are locally convex, then F is convex in U .

Thus, by Lemma 4.11 (1), the differential 1-form μ is closed in $\mathbb{R}_{>0}^6$.

For each subset S of the edges, by Equation (4.7), Lemma 4.7 and a direct calculation, $\mu|_{\mathcal{D}_S}$ is closed in \mathcal{L}_S . By definition, $\mu|_{\mathcal{D}_S}$ is constant in each connected component of $\mathcal{D}_S \setminus \mathcal{L}_S$. Now Proposition 4.8 shows that the subset \mathcal{L}_S in \mathcal{D}_S is open and bounded by a smooth codimension-1 submanifold. Thus, by Lemma 4.11 (1), the differential 1-form $\mu|_{\mathcal{D}_S}$ is closed in \mathcal{D}_S . For each Euclidean triangle Δ in a quadrant Q of \mathbb{R}^6 , let S be set of edges e so that $l(e) > 0$ for all $l \in Q$, and let Δ_S be the projection of Δ to \mathcal{D}_S . By definition and Lemma 4.7, $a_{ij}(l) \equiv 0$ if $l_{ij} \leq 0$, which implies $\int_{\Delta} \mu = \int_{\Delta_S} \mu|_{\mathcal{D}_S} = 0$. As a consequence, μ is closed in each of the quadrants of \mathbb{R}^6 . Repeating applying Lemma 4.11 (1), we conclude that μ is closed in \mathbb{R}^6 . \square

Corollary 4.12. The function $cov: \mathbb{R}^6 \rightarrow \mathbb{R}$ defined by the integral

$$cov(l) = \int_{(0, \dots, 0)}^l \mu + cov(0, \dots, 0) \quad (4.8)$$

is a C^1 -smooth convex function.

Proof. By Lemma 4.11 (2) and Proposition 4.10, cov is a C^1 -smooth function in \mathbb{R}^6 . By Lemma 4.11 (3) and Lemma 4.9, cov is convex in $\mathbb{R}_{>0}^6$. By the continuity, for each subset S of the edges, $cov|_{\mathcal{D}_S}$ is convex in \mathcal{D}_S . Since $a_{ij}(l) \equiv 0$ if $l_{ij} \leq 0$, we have $cov(l) = cov(l^+)$ and $l^+ \in \mathcal{D}_S$ for some S . As a consequence, cov is convex in each quadrant of \mathbb{R}^6 . Repeat using Lemma 4.11 (3), we conclude that cov is convex in \mathbb{R}^6 . \square

Remark 4.13. By the work of Ushijima [39], $cov(0, \dots, 0) = 2vol(0, \dots, 0) = 16\Lambda(\pi/4)$, where Λ is the Lobachevsky function defined by

$$\Lambda(a) = - \int_0^a \ln |2 \sin t| dt,$$

and $vol(0, \dots, 0)$ is the maximal volume amongst the generalized hyperbolic tetrahedra.

5 Global rigidity of hyper-ideal polyhedral metrics

In this section, we prove Theorem 1.2 (b) using the convex extension of cov in corollary 4.12.

Let (M, \mathcal{T}) be triangulated closed pseudo 3-manifold and let $E = E(\mathcal{T})$, $V = V(\mathcal{T})$ and $T = T(\mathcal{T})$ respectively be the sets of edges, vertices and tetrahedra in \mathcal{T} . Replacing each 3-simplex in \mathcal{T} by a hyper-ideal tetrahedron and gluing them along codimension-1 by isometries, we obtain an *hyper-ideal polyhedral metric* on (M, \mathcal{T}) . This metric is the same as assigning a positive number to each edge $e \in E(\mathcal{T})$ so that each tetrahedron σ becomes a hyper-ideal tetrahedron with assigned numbers as edge lengths. They are the same as hyperbolic cone metrics on $M - N(V)$ with singularity consisting of geodesic arcs between totally geodesic boundary. Here $N(V)$ is an open regular neighborhood of V in M . We denote by $\mathcal{L}(M, \mathcal{T})$ the space of all hyper-ideal polyhedral metrics on (M, \mathcal{T}) parametrized by the edge length vector $l: E \rightarrow \mathbb{R}_{>0}$. If $l \in \mathcal{L}(M, \mathcal{T})$, its *curvature* is a map $K_l: E(\mathcal{T}) \rightarrow \mathbb{R}$ sending each edge to 2π less the sum of dihedral angles at the edge. The *curvature map* $K: \mathcal{L}(M, \mathcal{T}) \rightarrow \mathbb{R}^E$ sends l to K_l .

The rigidity theorem 1.2 (b) can be rephrased as

Theorem 5.1. *For any closed triangulated pseudo 3-manifold (M, \mathcal{T}) , a hyper-ideal polyhedral metric on (M, \mathcal{T}) is determined by its curvature, i.e., the curvature map $K: \mathcal{L}(M, \mathcal{T}) \rightarrow \mathbb{R}^{E(\mathcal{T})}$ is injective.*

Proof. For each $l \in \mathcal{L}(M, \mathcal{T})$ and each tetrahedron $\sigma \in T$, let $l_\sigma \in \mathbb{R}_{>0}^6$ be the edge length vector of σ in the hyperbolic polyhedral metric l . Define the *co-volume* function

$$cov(l) = \sum_{\sigma \in T} cov(l_\sigma) \quad (5.1)$$

on $\mathcal{L}(M, \mathcal{T})$. By proposition 4.2 (a), the Hessian matrix of the function $cov(l_\sigma)$ is positive definite at each point in \mathcal{L} . Therefore, the Hessian matrix of cov is positive definite. In particular, cov is locally strictly convex. On the other hand, it is shown in [21] that the gradient of cov is 2π minus the curvature map $K_l \in \mathbb{R}^E$, i.e., $\nabla cov = 2\pi(1, 1, \dots, 1) - K_l$. Therefore, it suffices to show that ∇cov is injective.

We extend the co-volume function $cov: \mathcal{L}(M, \mathcal{T}) \rightarrow \mathbb{R}$ defined by (5.1) to a C^1 -smooth convex function, still denoted by $cov: \mathbb{R}_{>0}^E \rightarrow \mathbb{R}$, by

$$cov(l) = \sum_{\sigma \in T} cov(l_\sigma),$$

where $cov(l_\sigma)$ is the extended convex function given by corollary 4.12. The convexity of cov follows from the fact that each summand $cov(l_\sigma)$ is convex.

Now suppose otherwise that there exist $l_1 \neq l_2 \in \mathcal{L}(M, \mathcal{T})$ so that $K_{l_1} = K_{l_2}$. Join l_1 and l_2 in $\mathbb{R}_{>0}^E$ by the line segment $tl_1 + (1-t)l_2$, $t \in [0, 1]$, and consider the convex function $w(t) = cov(tl_1 + (1-t)l_2)$, $t \in [0, 1]$. By the construction, $w: [0, 1] \rightarrow \mathbb{R}$ is a C^1 -smooth convex function so that $w'(t) = \nabla cov \cdot (l_2 - l_1)$. Now $\nabla cov(l_i) = (2\pi, \dots, 2\pi) - K_{l_i}$ and $K_{l_1} = K_{l_2}$. It follows that $w'(0) = w'(1)$. Since w is convex, $w(t)$ must be a linear function in t . However, cov is strictly convex near l_1 and l_2 . Therefore, w is strictly convex in t near 0 and 1. This is a contradiction. \square

6 Volume maximization of angle structures

Let (M, \mathcal{T}) be closed triangulated pseudo 3-manifold with set of edges E and set of tetrahedra T .

Definition 6.1 (Angle assignment of hyper-ideal type). *An angle assignment (respectively positive angle assignment) of hyper-ideal type on (M, \mathcal{T}) assigns each edge e in each tetrahedra σ a non-negative (respectively positive) number $a(e, \sigma)$, call the dihedral angle of e in σ , so that sum of the dihedral angles at three edges in the same tetrahedron σ adjacent to each vertex is less than or equal to (respectively strictly less than) π . The cone angle of an angle assignment is the function $k \in \mathbb{R}_{\geq 0}^E$ sending each edge e to the sum of dihedral angles at e .*

For any $k \in \mathbb{R}_{\geq 0}^E$, we denote respectively by $\mathcal{B}_k^*(M, \mathcal{T})$ and $\mathcal{B}_k(M, \mathcal{T})$ the spaces of angle assignments and positive angle assignments with cone angles k . Note that if $\mathcal{B}_k \neq \emptyset$, then \mathcal{B}_k^* is the compact closure of \mathcal{B}_k . When $k = (2\pi, \dots, 2\pi)$, we denote respectively by $\mathcal{B}^*(M, \mathcal{T})$ and $\mathcal{B}(M, \mathcal{T})$ the space of the corresponding angle assignments and positive angle assignments of cone angles 2π . We note that $\mathcal{B}(M, \mathcal{T})$ coincides with the space of *linear hyperbolic structures* on (M, \mathcal{T}) defined in [21].

By the work of [2] and [14], a positive angle assignment a on (M, \mathcal{T}) is the same as making each tetrahedra $\sigma \in T$ a hyper-ideal tetrahedron so that its dihedral angles are given by a . In particular, we can define the volume of a positive angle assignment as the sum of the hyperbolic volume of the hyper-ideal tetrahedra, i.e., the *volume function* $vol: \mathcal{B}_k(M, \mathcal{T}) \rightarrow \mathbb{R}$ is

$$vol(a) = \sum_{\sigma \in T} vol(a_\sigma),$$

where $a_\sigma \in \mathcal{B}$ is the hyper-ideal tetrahedron with dihedral angles given by a . By the work of Schlenker [36], vol is smooth and strictly convex on $\mathcal{B}_k(M, \mathcal{T})$. A theorem of Rivin [35] shows that vol can be extended continuously to the compact closure $\mathcal{B}_k^*(M, \mathcal{T})$ of $\mathcal{B}_k(M, \mathcal{T})$.

Definition 6.2 (Generalized hyper-ideal metric). *We call a map $l: E \rightarrow \mathbb{R}_{>0}$ a generalized hyper-ideal metric on (M, \mathcal{T}) .*

For each $l \in \mathbb{R}_{>0}^E$, by replacing each tetrahedron σ in T by a generalized hyper-ideal tetrahedron whose edge lengths are given by l , we define the *dihedral angle* of l at an edge e in a tetrahedron $\sigma > e$ to be the corresponding dihedral angles in the generalized hyper-ideal tetrahedron whose edge lengths are given by l . The *cone angle* of l at e is the sum of dihedral angles of l at e in all the tetrahedra σ that contain e . The goal of this section is to prove the following

Theorem 6.3. *Suppose (M, \mathcal{T}) is a closed triangulated pseudo 3-manifold so that $\mathcal{B}_k(M, \mathcal{T}) \neq \emptyset$. Then,*

- (a) *there is a unique $a \in \mathcal{B}_k^*(M, \mathcal{T})$ that achieves the maximum volume, and*
- (b) *for each generalized hyper-ideal metric $l \in \mathbb{R}_{>0}^E$ with cone angles k , the dihedral angles $a(l)$ of l is the maximum volume on $\mathcal{B}_k^*(M, \mathcal{T})$.*
- (c) *If $a \in \mathcal{B}_k^*(M, \mathcal{T})$ achieves the maximum volume, then there exists a generalized hyperbolic metric $l \in \mathbb{R}_{>0}^E$ whose dihedral angles $a(l) = a$.*

Theorem 6.3 implies Theorem 1.4 in §1. It is known to Kojima [19] that every compact hyperbolic 3-manifold with totally geodesic boundary admits a geometric ideal triangulation so that each tetrahedron is either hyper-ideal or flat hyper-ideal whose dihedral angles are 0 and π . One consequence of Theorem 6.3 (b) is

Corollary 6.4. *Suppose M is a compact hyperbolic 3-manifold with totally geodesic boundary and \mathcal{T} is a geometric ideal triangulation of M so that each tetrahedron is either hyper-ideal or flat hyper-ideal. If $\mathcal{B}(M, \mathcal{T}) \neq \emptyset$, then the maximum volume on $\mathcal{B}^*(M, \mathcal{T})$ is equal to the hyperbolic volume of M .*

The counterpart of Corollary 6.4 for cusped hyperbolic 3-manifolds was proved in [24].

6.1 Volume function and sub-derivatives

Recall that $\mathcal{B} \subset \mathbb{R}^6$ is the space of dihedral angle vectors of hyper-ideal tetrahedra defined by (4.1) and let $\overline{\mathcal{B}}$ be its closure, i.e.,

$$\overline{\mathcal{B}} = \{(a_{12}, \dots, a_{34}) \in \mathbb{R}_{\geq 0}^6 \mid \sum_{j \neq i} a_{ij} \leq \pi \text{ for each } i, \text{ where } a_{ij} = a_{ji}\}.$$

By [35], the volume function vol on $\overline{\mathcal{B}}$ is continuous and convex. To study the volume optimization, we need to classify points in $\overline{\mathcal{B}}$.

Definition 6.5. We call $a = (a_{12}, \dots, a_{34}) \in \overline{\mathcal{B}}$ *generalized dihedral angles*

- I. of type I if for each $i \in \{1, \dots, 4\}$, $\sum_{j \neq i} a_{ij} < \pi$,
- II. of type II if $a = (\pi, 0, 0, 0, 0, \pi)$, $(0, \pi, 0, 0, \pi, 0)$ or $(0, 0, \pi, \pi, 0, 0)$, and
- III. of type III if not of types I and II.

We denote by \mathcal{B}_I , \mathcal{B}_{II} and \mathcal{B}_{III} respectively the set of generalized dihedral angles of types I, II and III. Note that $\mathcal{B} \subset \mathcal{B}_I$. For each $\{i, j\} \subset \{1, \dots, 4\}$, we let $\psi_{ij} : \mathcal{B} \rightarrow \mathbb{R}$ be the function defined by

$$\psi_{ij}(a) = \frac{s_{ij}^2 c_{kh} + c_{ik} c_{jk} + c_{ih} c_{jh} + c_{ij} c_{ik} c_{jh} + c_{ij} c_{ih} c_{jk}}{\sqrt{2c_{ij} c_{ik} c_{ih} + c_{ij}^2 + c_{ik}^2 + c_{ih}^2 - 1} \sqrt{2c_{ij} c_{jk} c_{jh} + c_{ij}^2 + c_{jk}^2 + c_{jh}^2 - 1}},$$

where $s_{ij} = \cos a_{ij}$ and $c_{ij} = \cos a_{ij}$. By the Cosine Law, if $a \in \mathcal{B}$ is the dihedral angles of a hyper-ideal tetrahedron σ with $l(a) = (l_{12}(a), \dots, l_{34}(a))$ the edge lengths, then $l_{ij}(a) = \cosh^{-1} \psi_{ij}(a)$.

Lemma 6.6. The function $\psi_{ij} : \mathcal{B} \rightarrow \mathbb{R}$ continuously extends to \mathcal{B}_I , and $\psi_{ij}(a) = 1$ when $a_{ij} = 0$.

Proof. Since $\sum_{j \neq i} a_{ij} < \pi$ for $i = 1, \dots, 4$, the denominator of ψ_{ij} is not equal to 0, hence the function continuously extends. If $a_{ij} = 0$, then $c_{ij} = 1$ and $\psi_{ij}(a) = 1$. \square

For $a \in \mathcal{B}_I$, we let $l_{ij}(a) = \cosh^{-1} \psi_{ij}(a)$ and call $l(a) = (l_{12}(a), \dots, l_{34}(a)) \in \mathbb{R}_{\geq 0}^6$ the *associated edge lengths* of a . Given a tetrahedron σ , let S be a set of edges in σ and define

$$\mathcal{B}_S \doteq \{a \in \mathcal{B}_I \mid a_\alpha > 0 \text{ for } \alpha \in S \text{ and } a_\alpha = 0 \text{ for } \alpha \notin S\}.$$

The set \mathcal{B}_S is an open convex polytope in the smallest affine space containing it since \mathcal{B}_S is defined by strict linear inequalities and linear equalities.

Proposition 6.7. For each S , the restriction of the volume function is smooth and strictly concave in \mathcal{B}_S .

Proof. By Lemma 6.6, the edge lengths continuously extend to \mathcal{B}_S with $l_\alpha = 0$ for $\alpha \notin S$. From the definition, each ψ_α for $\alpha \in S$ is smooth in \mathcal{B}_S . As a consequence, the function $l_\alpha = \cosh^{-1} \psi_\alpha$ is smooth and the following differential 1-form $\omega_S = -\frac{1}{2} \sum_{\alpha \in S} l_\alpha da_\alpha$ is smooth in \mathcal{B}_S . From the definition and a direct calculation, we have $\frac{\partial l_\alpha}{\partial a_\beta} = \frac{\partial l_\beta}{\partial a_\alpha}$ for $\alpha, \beta \in S$, i.e., the differential 1-form ω_S is closed in \mathcal{B}_S . For each $a \in \mathcal{B}_S$, take $a_0 \in \mathcal{B}_S$ close to a , and define a function $g : \mathcal{B}_S \rightarrow \mathbb{R}$ by $g(a) = \int_{a_0}^a \omega_S$, where the path in the integral is any smooth path in \mathcal{B}_S connecting a_0 and a . Since ω_S is smooth and closed, g is well defined and smooth in \mathcal{B}_S and $\frac{\partial g}{\partial a_\alpha} = -\frac{l_\alpha}{2}$ for each corner $\alpha \in S$. We claim that $g(a) = vol(a) - vol(a_0)$. The claim implies that the volume function vol is smooth in \mathcal{B}_S and $\frac{\partial vol}{\partial a_\alpha} = -\frac{l_\alpha}{2}$ for each $\alpha \in S$. Now since a_0 is close to a , we can take a vector $w \in \mathbb{R}^6$ so that $a + w, a_0 + w \in \mathcal{B}$. Let

$v = a - a_0$ and let $f(t, s) \doteq -\frac{1}{2} \sum_{\alpha} l_{\alpha}(a_0 + tv + sw)v_{\alpha}$, where the summation is over all the dihedral angles. By the continuity of vol , we have $\text{vol}(a) - \text{vol}(a_0) = \lim_{s \rightarrow 0^+} (\text{vol}(a + sw) - \text{vol}(a_0 + sw)) = \lim_{s \rightarrow 0^+} \int_0^1 f(t, s) dt$. On the other hand, since l_{α} continuously extends to \mathcal{B}_S , it is uniformly continuous on the compact parallelogram determined by $a_0, a, a_0 + w$ and $a + w$. As a consequence, f is uniformly continuous on the compact square $[0, 1] \times [0, 1]$ and $g(a) = \int_0^1 f(t, 0) dt = \int_0^1 \lim_{s \rightarrow 0^+} f(t, s) dt$. Again, since f is uniformly continuous, we can switch the order of taking limit and integrating, and we have $g(a) = \text{vol}(a) - \text{vol}(a_0)$.

Now we show the strict concavity of vol in \mathcal{B}_S . Let $l : \mathcal{B}_S \rightarrow \mathbb{R}_{\geq 0}^6$ be map defined by the restriction of the associated edge lengths and let $\mathcal{L}_S = l(\mathcal{B}_S)$. From the definition, each ϕ_{α} for $\alpha \in S$ is smooth in \mathcal{L}_S , hence the function $a_{\alpha} = \cos^{-1} \phi_{\alpha}$ is smooth in \mathcal{L}_S . Now we have two smooth maps $a = (a_{\alpha}) : \mathcal{L}_S \rightarrow \mathcal{B}_S$ and $l = (l_{\alpha}) : \mathcal{B}_S \rightarrow \mathcal{L}_S$ which are, by the Cosine Law, inverses of each other. As a consequence, the map $l : \mathcal{B}_S \rightarrow \mathcal{L}_S$ is a local diffeomorphism and the Jacobi matrix $[\partial l_{\alpha} / \partial a_{\beta}]_{\alpha, \beta \in S}$ is non-singular. Since \mathcal{B}_S is connected, the signature of the Hessian matrix $-\frac{1}{2} [\partial l_{\alpha} / \partial a_{\beta}]_{\alpha, \beta \in S}$ of the volume function vol on \mathcal{B}_S is independent of the choice of $a \in \mathcal{B}_S$. By a direct calculation, using the formula in Guo ([16], Theorem 1), the matrix $-\frac{1}{2} [\partial l_{\alpha} / \partial a_{\beta}]_{\alpha, \beta \in S}$ is negative definite at $a_{\alpha} = \pi/4$ for each $\alpha \in S$. This implies that vol is locally strictly concave in \mathcal{B}_S . Since \mathcal{B}_S is convex, the volume function vol is strictly concave in \mathcal{B}_S . \square

Lemma 6.8. *Let $l \in \mathbb{R}_{>0}^6$ be the edge length vector of a generalized hyper-ideal tetrahedron with dihedral angles $a(l) \in \mathcal{B}_{II}$, and let $v \in \mathbb{R}^6$ so that $a(l) + v \in \mathcal{B}$. Then*

$$\lim_{t \rightarrow 0^+} \frac{d}{dt} \text{vol}(a(l) + tv) \leq -\frac{1}{2} v \cdot l.$$

Proof. Let $f : [0, 1] \rightarrow \mathbb{R}$ be the function defined by $f(t) = \text{vol}(a(l) + tv)$. For (1), by the concavity of vol and the Mean Value Theorem, we have $f'(t) < \frac{f(t) - f(0)}{t}$ for all $t \in (0, 1)$. Since $a(l) \in \mathcal{B}_{II}$, $f(0) = \text{vol}(a(l)) = 0$, and

$$f'(t) < \frac{f(t)}{t}. \quad (6.1)$$

By Proposition 4.5, \mathcal{L} is open in $\mathbb{R}_{>0}^6$ and $\partial \mathcal{L}$ is a smooth codimension-1 submanifold, hence for each $l \in \partial \mathcal{L}$, there exists a sequence $\{l^{(n)}\} \subset \mathcal{L}$ converging to l and the corresponding dihedral angles $\{a^{(n)}\}$ converging to $a(l)$. Since the volume function vol is strictly concave in \mathcal{B} , we have for all n and for all $b \in \mathcal{B}$ that $\text{vol}(b) - \text{vol}(a^{(n)}) < \nabla \text{vol}(a^{(n)}) \cdot (b - a^{(n)})$. By Schlaefli formula, $\text{vol}(b) - \text{vol}(a^{(n)}) < -\frac{l^{(n)}}{2} \cdot (b - a^{(n)})$. Since vol is continuous on $\overline{\mathcal{B}}$ and $\text{vol}(a(l)) = 0$, as n approaches $+\infty$, $\text{vol}(b) \leq -\frac{l}{2} \cdot (b - a(l))$. In particular, when $b = a(l) + tv$, we have

$$f(t) \leq -\frac{t}{2} v \cdot l. \quad (6.2)$$

By (6.1) and (6.2), we have $f'(t) < -\frac{1}{2} v \cdot l$, hence $\lim_{t \rightarrow 0^+} f'(t) \leq -\frac{1}{2} v \cdot l$. For $l = (l_{12}, \dots, l_{34}) \notin \overline{\mathcal{L}}$, by Proposition 4.4, there exists an $m = (m_{12}, l_{13}, \dots, l_{34}) \in \partial \mathcal{L}$ such that $a(m) = a(l)$. By the previous case, we have

$$\lim_{t \rightarrow 0^+} f'(t) \leq -\frac{1}{2} v \cdot m. \quad (6.3)$$

Without loss of generality, we may assume that $a(l) = (\pi, 0, 0, 0, 0, \pi)$. By Cosine Law, $l_{12} > m_{12}$. Since $a(l) + v \in \mathcal{B}$, we have $\pi + v_{12} = (a(l) + v)_{12} < \pi$, hence $v_{12} < 0$. As a consequence, $v \cdot l - v \cdot m = v_{12}(l_{12} - m_{12}) < 0$. Combined with (6.3), we have $\lim_{t \rightarrow 0^+} f'(t) < -\frac{1}{2} v \cdot l$. \square

Lemma 6.9. Let $a \in \mathcal{B}_{III}$. Then for each $v \in \mathbb{R}^6$ so that $a + v \in \mathcal{B}$, one has

$$\lim_{t \rightarrow 0^+} \frac{d}{dt} \text{vol}(a + tv) = +\infty.$$

Proof. We let $s_{ij}(t) = \sin(a_{ij} + tv_{ij})$, $c_{ij}(t) = \cos(a_{ij} + tv_{ij})$ and

$$u_{ij}(t) = s_{ij}^2(t)c_{kh}(t) + c_{ik}(t)c_{ih}(t) + c_{jk}(t)c_{jh}(t) + c_{ij}(t)c_{ik}(t)c_{jh}(t) + c_{ij}(t)c_{ih}(t)c_{jk}(t)$$

for $\{i, j\} \subset \{1, \dots, 4\}$, and let

$$b_i(t) = \sqrt{2c_{ij}(t)c_{ik}(t)c_{ih}(t) + c_{ij}^2(t) + c_{ik}^2(t) + c_{ih}^2(t) - 1}$$

for $i \in \{1, \dots, 4\}$. Then $\cosh l_{ij}(a + tv) = \frac{u_{ij}(t)}{b_i(t)b_j(t)}$. Let $m_{ij} \geq 0$ and $n_i \geq 0$ respectively be the asymptotic orders of $u_{ij}(t)$ and $b_i(t)$ as t approaches 0, i.e., $u_{ij}(t) = u_{ij}t^{m_{ij}} + o(t^{m_{ij}})$ and $b_i(t) = b_it^{n_i} + o(t^{n_i})$ for some constants $u_{ij} \neq 0$ and $b_i \neq 0$. As t approaches 0, $\cosh(l_{ij}(a + tv)) = \frac{u_{ij}}{b_ib_j}t^{m_{ij}-n_i-n_j} + o(t^{m_{ij}-n_i-n_j})$, and by Schlaefli formula,

$$\begin{aligned} & \lim_{t \rightarrow 0^+} \left(\frac{d}{dt} \text{vol}(a + tv) - \left(-\frac{1}{2} \sum_{i \neq j} v_{ij}(m_{ij} - n_i - n_j) \right) \ln t \right) \\ &= -\frac{1}{2} \sum_{i \neq j} v_{ij} \lim_{t \rightarrow 0} \left(l_{ij}(a + tv) - (m_{ij} - n_i - n_j) \ln t \right) = -\frac{1}{2} \sum_{i \neq j} v_{ij} \ln \frac{u_{ij}}{b_ib_j} \end{aligned}$$

is a finite number. Therefore, to prove the result, it suffices to prove that the $-\frac{1}{2} \sum_{i \neq j} v_{ij}(m_{ij} - n_i - n_j)$ is strictly negative. First we consider the case that, up to a permutation of the vertices, $a \neq (0, \alpha, \pi - \alpha, \pi - \alpha, \alpha, 0)$ for some $a \in (0, \pi)$ and $a \neq (\pi, 0, 0, 0, 0, \beta)$ for some $\beta \in [0, \pi)$. In this case, if $a_{ij} = \pi$ for some $\{i, j\} \subset \{1, \dots, 4\}$, then $a_{ik} = a_{ih} = a_{jk} = a_{jh} = 0$, which was ruled out by the assumption. If $a_{ij} \in (0, \pi)$ for some $\{i, j\} \subset \{1, \dots, 4\}$, then we claim that $m_{ij} = 0$, i.e., $u_{ij}(0) \neq 0$. Indeed, letting $s_{ij} = \sin a_{ij}$ and $c_{ij} = \cos a_{ij}$, we have $u_{ij}(0) = s_{ij}^2(c_{kh} + c_{ik}c_{jk}) + (c_{ih} + c_{ij}c_{ik})(c_{jh} + c_{ij}c_{jk})$ and also $u_{ij}(0) = s_{ij}^2(c_{kh} + c_{ih}c_{jh}) + (c_{ik} + c_{ij}c_{ih})(c_{jk} + c_{ij}c_{jh})$. Since $a \in \bar{\mathcal{B}}$, we have $c_{pq} + c_{pr}c_{ps} \geq 0$, $\{p, q, r, s\} = \{1, 2, 3, 4\}$, and the equality holds if and only if $a_{pq} + a_{pr} + a_{ps} = \pi$ and one of a_{pr} and a_{ps} equals 0. Therefore, since $s_{ij}^2 > 0$, $u_{ij}(0) = 0$ only if $c_{kh} + c_{ik}c_{jk} = 0$, $c_{kh} + c_{ih}c_{jh} = 0$ and one of $c_{ih} + c_{ij}c_{ik}$ and $c_{jh} + c_{ij}c_{jk}$ equals 0. If, say, $c_{jh} + c_{ij}c_{jk} = 0$, then from the last equation, we have $a_{jk} = 0$ and $a_{jh} = \pi - a_{ij} \in (0, \pi)$. With $c_{kh} + c_{ih}c_{jh} = 0$, we have $a_{ih} = 0$ and $a_{kh} = \pi - a_{jh} = a_{ij}$. From $c_{kh} + c_{ik}c_{jk} = 0$ and $a_{jk} = 0$, we have $a_{ik} = \pi - a_{kh} = \pi - a_{ij}$. As a consequence, up to a permutation of indices, $a = (0, a_{ij}, \pi - a_{ij}, \pi - a_{ij}, a_{ij}, 0)$ with $a_{ij} \in (0, \pi)$, which was ruled out by the assumption. Hence the claim is true. Now for $a_{ij} = 0$, since $a + v \in \mathcal{B}$, we have $v_{ij} = a_{ij} + v_{ij} > 0$ and $-\frac{1}{2}v_{ij}(m_{ij} - n_i - n_j) \leq \frac{1}{2}v_{ij}(n_i + n_j)$. By the definition of $b_i(t)$, we have $n_i > 0$ if and only if $\sum_{j \neq i} a_{ij} = \pi$. Since a is of type III, there exists at least one $i \in \{1, \dots, 4\}$ with $\sum_{j \neq i} a_{ij} = \pi$. For such i , since $a + v \in \mathcal{B}$, we have $\sum_{j \neq i} v_{ij} < 0$. Thus, $-\frac{1}{2} \sum_{i \neq j} v_{ij}(m_{ij} - n_i - n_j) \leq \frac{1}{2} \sum_{i \neq j} v_{ij}(n_i + n_j) = \frac{1}{2} \sum_{i=1}^4 (\sum_{j \neq i} v_{ij})n_i < 0$. The two sporadic cases are verified by a direct calculation. In the case that $a = (0, \alpha, \pi - \alpha, \pi - \alpha, \alpha, 0)$ with $a \in (0, \pi)$, we have $n_i = 2$ for each $i \in \{1, \dots, 4\}$, $n_{13} = n_{14} = n_{23} = n_{24} = 1$ and $n_{12} = n_{34} = 2$. Since $a_{12} = a_{34} = 0$ and $\sum a_{ij} = 2\pi$, we have $v_{12} > 0$, $v_{34} > 0$ and $\sum v_{ij} < 0$. Therefore, $-\frac{1}{2} \sum v_{ij}(m_{ij} - n_i - n_j) = v_{12} + v_{34} + \frac{3}{2}(v_{13} + v_{14} + v_{23} + v_{24}) < \frac{3}{2} \sum v_{ij} < 0$. In the case that $a = (\pi, 0, 0, 0, 0, \beta)$ with $\beta \in [0, \pi)$, we have $n_1 = n_2 = 2$, $n_3 = n_4 = 0$, $m_{12} = m_{13} = m_{14} = m_{23} = m_{24} = 2$ and $m_{34} = 0$. Since $a_{12} = \pi$, we have $v_{12} < 0$ and $-\frac{1}{2} \sum v_{ij}(m_{ij} - n_i - n_j) = v_{12} < 0$. \square

6.2 Fenchel duality

Let (M, \mathcal{T}) be closed triangulated pseudo 3-manifold. The space of all generalized hyper-ideal polyhedral metrics on (M, \mathcal{T}) , parameterized by the edge length vectors, is \mathbb{R}^E . The *co-volume* function $cov: \mathbb{R}^E \rightarrow \mathbb{R}$ is defined by

$$cov(l) = \sum_{\sigma \in T} cov(l_\sigma),$$

where each summand is the function defined by (4.8). By Corollary 4.12, cov is C^1 -smooth and convex, hence its Fenchel dual function

$$cov^*(k) = \sup\{k \cdot l - cov(l) \mid l \in \mathbb{R}^E\}$$

is well-defined, convex and lower semicontinuous in \mathbb{R}^E . The goal of this subsection is to show that the Fenchel dual function $cov^*(k)$ optimizes the volume function on the space of non-positive angle assignments of given cone angles.

Theorem 6.10. *Let*

$$\mathcal{D}(\mathcal{T}) = \{k \in \mathbb{R}^E \mid \mathcal{B}_k^*(M, \mathcal{T}) \neq \emptyset\}$$

and let $U: \mathbb{R}^E \rightarrow \mathbb{R}$ be the function defined by

$$U(k) = \begin{cases} \min\{-2vol(a) \mid a \in \mathcal{B}_k^*(M, \mathcal{T})\} & \text{if } k \in \mathcal{D}(\mathcal{T}), \\ +\infty & \text{if } k \notin \mathcal{D}(\mathcal{T}). \end{cases}$$

Then $cov^(k) = U(k)$ for all $k \in \mathbb{R}^E$.*

The proof of Theorem 6.10 relies on the following propositions.

Proposition 6.11. *U is convex and continuous in $\mathcal{D}(\mathcal{T})$.*

Proof. The proof follows by the same argument of Proposition 3.5. □

Proposition 6.12. *For each $k \in \mathbb{R}^E$ so that $\mathcal{B}_k(M, \mathcal{T}) \neq \emptyset$, the function $cov_k: \mathbb{R}^E \rightarrow \mathbb{R}$ defined by*

$$cov_k(l) = cov(l) - k \cdot l$$

has a critical point. Moreover, all the critical points of cov_k are in $\mathbb{R}_{>0}^E$.

Proof. We will use the method developed by Colin de Verdière's [8]. Take any $a \in \mathcal{B}_k(M, \mathcal{T})$. We rewrite cov_k as

$$cov_k(l) = \sum_{\sigma \in T} (cov(l_\sigma) - a_\sigma \cdot l_\sigma).$$

For each $\sigma \in T$, we let $l(a_\sigma) \in \mathbb{R}^6$ be the edge length vector of the hyper-ideal tetrahedron whose dihedral angles are a_σ . Then $l(a_\sigma)$ is the unique critical point of the convex function $cov_{\sigma,k}: \mathbb{R}^6 \rightarrow \mathbb{R}$ defined by

$$cov_{\sigma,k}(l) = cov(l) - a_\sigma \cdot l.$$

Since $cov_{\sigma,k}$ is strictly convex near l_σ , the function $cov_{\sigma,k}$ is *closed* in \mathbb{R}^6 , i.e., $\lim_{|l| \rightarrow +\infty} cov_{\sigma,k}(l) = +\infty$. As a consequence, the function cov_k is closed and convex in \mathbb{R}^E . This shows that cov_k has a critical point l in \mathbb{R}^E . Moreover, since $\mathcal{B}_k(M, \mathcal{T}) \neq \emptyset$, $k(e) > 0$ for each edge e . This implies that $l(e) > 0$ for each $e \in E$, i.e., $l \in \mathbb{R}_{>0}^E$. Indeed, if otherwise $l(e) \leq 0$ for some $e \in E$, then by Lemma 4.7, all the dihedral angles of l at e are zero, hence $k(e) = 0$, which is a contradiction. □

As consequences of Proposition 6.12, we have

Proposition 6.13. *If $\mathcal{B}_k(M, \mathcal{T}) \neq \emptyset$, then there exists a generalized hyper-ideal metric $l \in \mathbb{R}_{>0}^E$ such that the dihedral angles $a(l) \in \mathcal{B}_k^*(M, \mathcal{T})$.*

Proof. Any critical point of cov_k satisfies the desired condition. \square

Proposition 6.14. *The image $K(\mathcal{L}(M, \mathcal{T})) \cap (\pi, 2\pi)^E$ is a convex open polytope in \mathbb{R}^E .*

Proof. Denoting by $\mathcal{K}(\mathcal{T})$ the convex open polytope

$$\{(2\pi, \dots, 2\pi) - k \mid \mathcal{B}_k(M, \mathcal{T}) \neq \emptyset\}.$$

We claim that $K(\mathcal{L}(M, \mathcal{T})) \cap (\pi, 2\pi)^E = \mathcal{K}(\mathcal{T}) \cap (\pi, 2\pi)^E$. Indeed, since $K(\mathcal{L}(M, \mathcal{T})) \subseteq \mathcal{K}(\mathcal{T})$, we have that $K(\mathcal{L}(M, \mathcal{T})) \cap (\pi, 2\pi)^E \subseteq \mathcal{K}(\mathcal{T}) \cap (\pi, 2\pi)^E$. On the other hand, by Proposition 6.13, for each $k \in \mathcal{K}(\mathcal{T}) \cap (\pi, 2\pi)^E$, there exists an $l \in \mathbb{R}_{>0}^E$ such that $K(l) = k$. It now suffices to show that $l \in \mathcal{L}(M, \mathcal{T})$. Since $k \in (\pi, 2\pi)^E$, the cone angle of l at each edge e is in the range $(0, \pi)$. As a consequence, all the dihedral angles of \mathcal{T} in l are in the range $(0, \pi)$. Thus, all the tetrahedra of \mathcal{T} are hyper-ideal in l , and l is in $\mathcal{L}(M, \mathcal{T})$. \square

There are examples showing that the whole image $K(\mathcal{L}(M, \mathcal{T}))$, or the subset $K(\mathcal{L}(M, \mathcal{T})) \cap (0, 2\pi)^E$, is in general neither convex nor a polytope in \mathbb{R}^E .

Proof of Theorem 6.10. We first show that if $k \notin \mathcal{D}(\mathcal{T})$, then $\text{cov}^*(k) > C$ for all $C > 0$. Since the space $\overline{\mathcal{B}}^T$ of all possible dihedral angles is compact and vol is continuous, there exists a constant $C_1 > 0$ so that $\text{vol}(a) \leq C_1$ for all $a \in \overline{\mathcal{B}}^T$. Since $\{k\}$ and $\mathcal{D}(\mathcal{T})$ are compact and convex in \mathbb{R}^E , by the Separation Theorem of Convex Sets, there exists an $l_0 \in \mathbb{R}^E$ so that

$$k \cdot l_0 - c \cdot l_0 > C + 2C_1$$

for all $c \in \mathcal{D}(\mathcal{T})$. In particular, letting $c(l_0) \in \mathcal{D}(\mathcal{T})$ be the cone angle vector of l_0 , we have

$$k \cdot l_0 - c(l_0) \cdot l_0 > C + 2C_1.$$

Therefore,

$$\begin{aligned} \text{cov}^*(k) &\geq k \cdot l_0 - \text{cov}(l_0) \\ &= k \cdot l_0 - c(l_0) \cdot l_0 - 2\text{vol}(a(l_0)) \\ &> C + 2C_1 - 2C_1 = C. \end{aligned}$$

Now we prove that $\text{cov}^*(k) = U(k)$ in $\mathcal{D}(\mathcal{T})$. By Proposition 6.12, if $\mathcal{B}_k(M, \mathcal{T}) \neq \emptyset$, the function cov_k has a critical point $l \in \mathbb{R}_{>0}^E$, and $\text{cov}^*(k) = -\text{cov}_k(l) = -2\text{vol}(a(l))$. To show that $U(k) = -2\text{vol}(a(l))$, i.e., $a(l)$ achieves the maximum volume, it suffices to show that the sub-derivative

$$\lim_{t \rightarrow 0^+} \frac{d}{dt} \text{vol}((1-t)a(l) + tb) \leq 0$$

for each $b \in \mathcal{B}_k(M, \mathcal{T})$. Let $v = b - a(l)$ and let $v_\sigma = b_\sigma - a_\sigma(l)$ for each $\sigma \in T$. We have $\sum_{\sigma \supset e} b(e, \sigma) = \sum_{\sigma \supset e} a(l)(e, \sigma) = k(e)$, hence $\sum_{\sigma \supset e} v(e, \sigma) = 0$ for each edge e . From this, we

see $\sum_{\sigma \in T} v_\sigma \cdot l_\sigma = \sum_{e \in E} \left(\sum_{\alpha < e} v_\alpha \right) l(e) = 0$. Let S be the subset of T consisting of hyper-ideal tetrahedra in l and let $F = T \setminus S$. We have $-\sum_{\sigma \in S} v_\sigma \cdot l_\sigma = \sum_{\sigma \in F} v_\sigma \cdot l_\sigma$. Then by Schläfli formula,

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{d}{dt} \text{vol}(a(l) + tv) &= \sum_{\sigma \in F} \lim_{t \rightarrow 0^+} \frac{d}{dt} \text{vol}(a_\sigma(l) + tv_\sigma) - \frac{1}{2} \sum_{\sigma \in S} v_\sigma \cdot l_\sigma \\ &= \sum_{\sigma \in F} \left(\lim_{t \rightarrow 0^+} \frac{d}{dt} \text{vol}(a_\sigma(l) + tv_\sigma) + \frac{1}{2} v_\sigma \cdot l_\sigma \right) \leq 0, \end{aligned}$$

where the inequality is from Lemma 6.8. Hence cov^* and U coincide on the subset $\{k \mid \mathcal{B}_k(M, \mathcal{T}) \neq \emptyset\}$ of $\mathcal{D}(\mathcal{T})$ which contains the relative interior of $\mathcal{D}(\mathcal{T})$. Since cov^* and U are convex and lower semicontinuous, by Lemma 3.9, $\text{cov}^*(k) = U(k)$ on $\mathcal{D}(\mathcal{T})$. \square

6.3 Proofs of the main results

6.3.1 A proof of Theorem 6.3

For (a), suppose otherwise that there exist $a^{(1)} \neq a^{(2)} \in \mathcal{B}_k^*(M, \mathcal{T})$ that achieve the maximum volume. Connect $a^{(1)}$ and $a^{(2)}$ by the line segment $L(t) = ta^{(1)} + (1-t)a^{(2)}$, $t \in [0, 1]$, and consider the concave function $f(t) = \text{vol}(ta^{(1)} + (1-t)a^{(2)})$. By the maximality of $a^{(i)}$, the function vol is constant in $[0, 1]$. On the other hand, we let $a_\sigma^{(i)} \in \bar{\mathcal{B}}$ be the restriction of $a^{(i)}$ to σ and let $f_\sigma(t) = \text{vol}(ta_\sigma^{(1)} + (1-t)a_\sigma^{(2)})$ for each $\sigma \in T$. Then by the maximality and Lemma 6.9, each $L_\sigma(t) \doteq ta_\sigma^{(1)} + (1-t)a_\sigma^{(2)}$ is not of type III. As a consequence, the interior of the line segment L_σ lies in \mathcal{B}_I or \mathcal{B}_{II} . Since \mathcal{B}_{II} is discrete in $\bar{\mathcal{B}}$ and $a^{(1)} \neq a^{(2)}$, there is at least one $\sigma_0 \in T$ such that L_{σ_0} lies in \mathcal{B}_I . We claim that the interior of L_{σ_0} lies in \mathcal{B}_S for some subset S of the edges of σ_0 . Indeed, if $L_{\sigma_0}(t_1) \in \mathcal{B}_{S_1}$ and $L_{\sigma_0}(t_2) \in \mathcal{B}_{S_2}$ for some $t_1, t_2 \in (0, 1)$, then the interior of L_{σ_0} lies in $\mathcal{B}_{S_1 \cup S_2}$. By Proposition 6.7, the function f_{σ_0} is strictly concave, hence $f = \sum_{\sigma \in T} f_\sigma$ is strictly concave in $[0, 1]$, which is a contradiction. For (b), by the assumption, l is a critical point of the function cov_k , and $\text{cov}_k(l) = 2\text{vol}(a(l))$. By Theorem 6.10, $\text{cov}_k(l) = -\text{cov}^*(k) = -U(k) = \max\{2\text{vol}(a) \mid a \in \mathcal{B}_k^*(M, \mathcal{T})\}$. For (c), by Proposition 6.13, there exists an $l \in \mathbb{R}_{>0}^E$ such that $a(l) \in \mathcal{B}_k^*(M, \mathcal{T})$. By (b), $a(l)$ achieves the maximum volume, and by (a), $a(l) = a$.

6.3.2 A proof of Corollary 6.4

Let $l \in \mathbb{R}_{>0}^E$ be the edge length vector in the hyperbolic metric of (M, \mathcal{T}) for which \mathcal{T} is geometric. Then the dihedral angles $a(l) \in \mathcal{B}^*(M, \mathcal{T})$. By Theorem 6.3 (b), $a(l)$ achieves the maximum volume on $\mathcal{B}^*(M, \mathcal{T})$. Since the triangulation \mathcal{T} is geometric, $\text{vol}(a(l))$ equals the hyperbolic volume of M .

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